Modeling the Volatility Smile

Emanuel Derman

Columbia University

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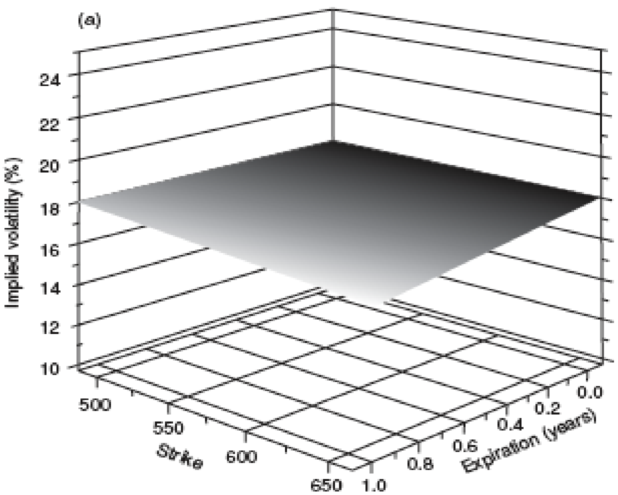
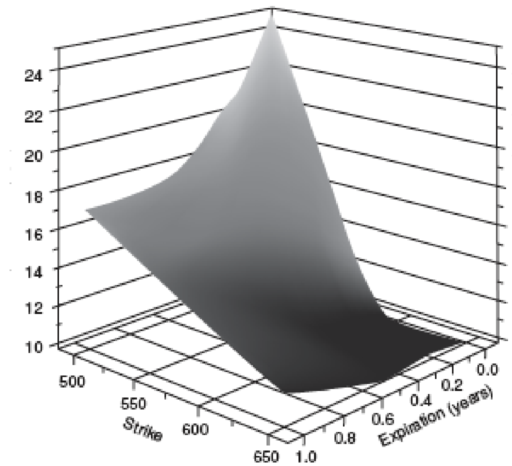
The Implied Volatility Smile/Surface

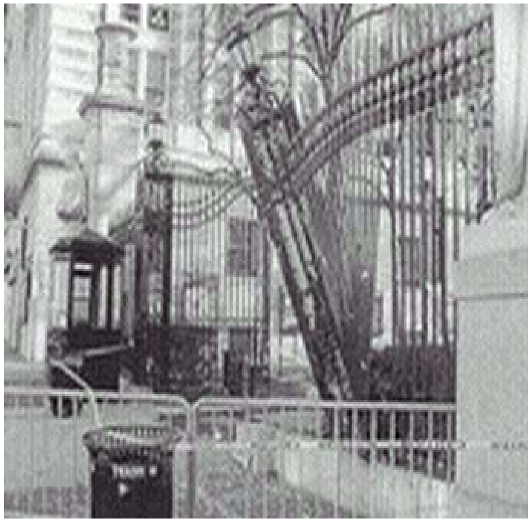


Black-Scholes



re 14.2 A typical implied volatility surface for the S&P 500 in mid-1995.

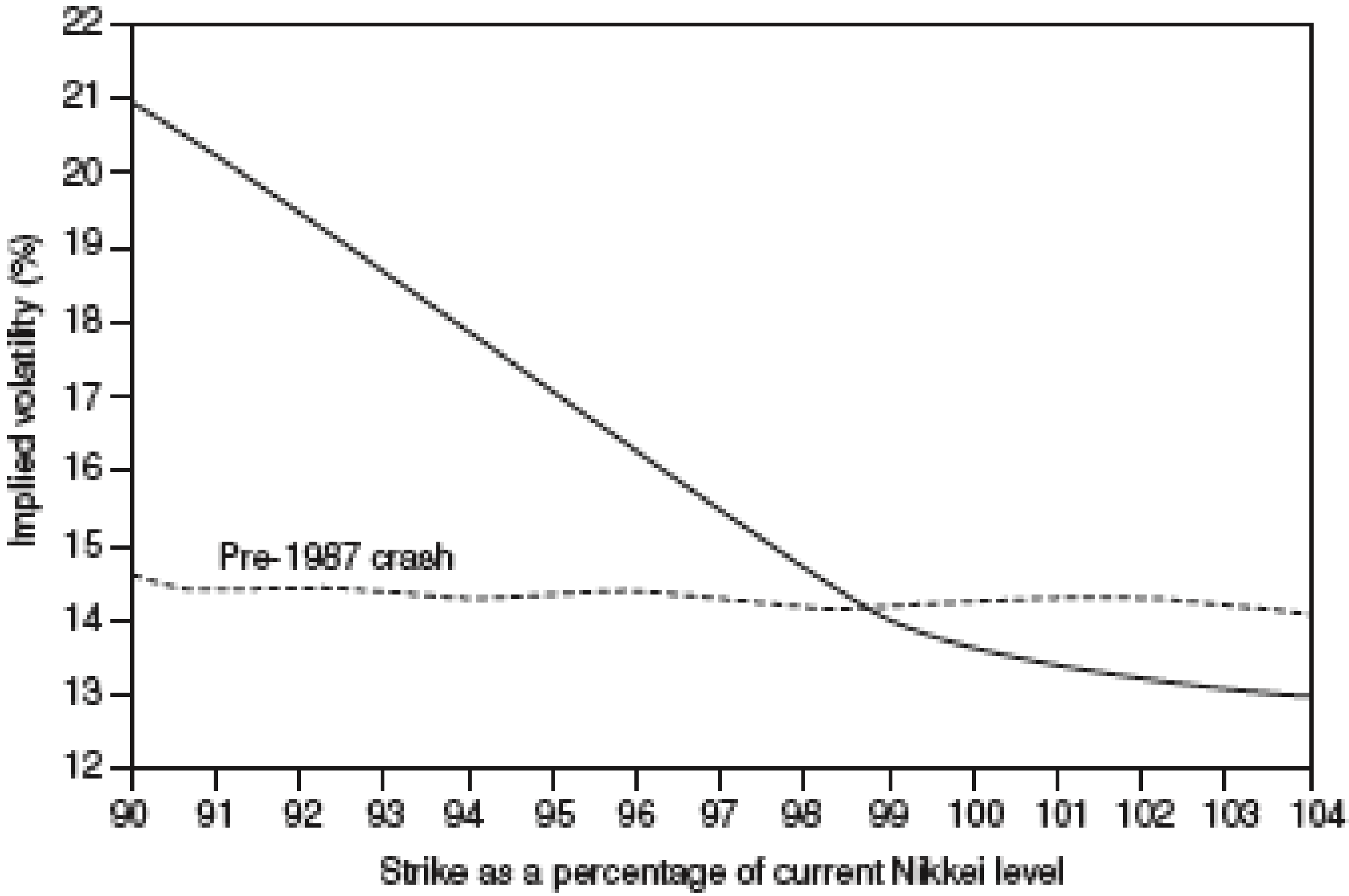
S&P 1995



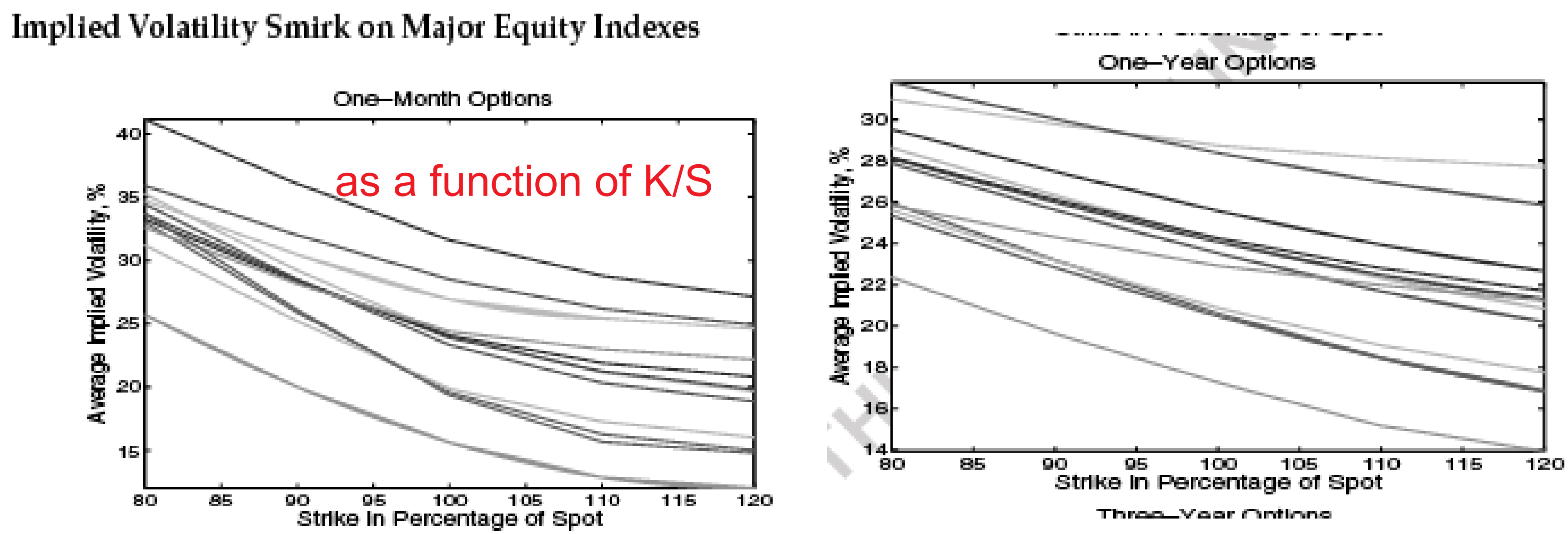
B way & 116th, 2004

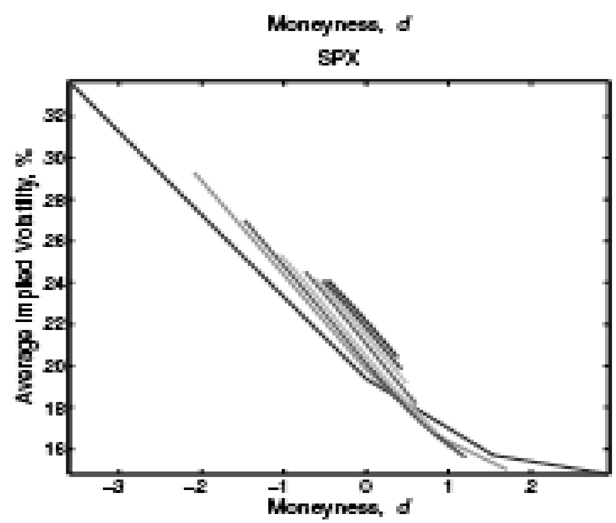
- Black-Scholes implied volatilities for equity indices: $\Sigma(S, t, K, T)$
- Term structure of strike and expiration, which change with time and market level
- Always a negative slope w.r.t strike for equity index options
- What model replaces Black-Scholes?

The Smile is a Post-Crash Phenomenon



Current Equity Index Smile





as a function of moneyness $\left[\frac{\ln(K/S)}{\sigma\sqrt{\tau}}\right]$ or Δ

it steepens with maturity

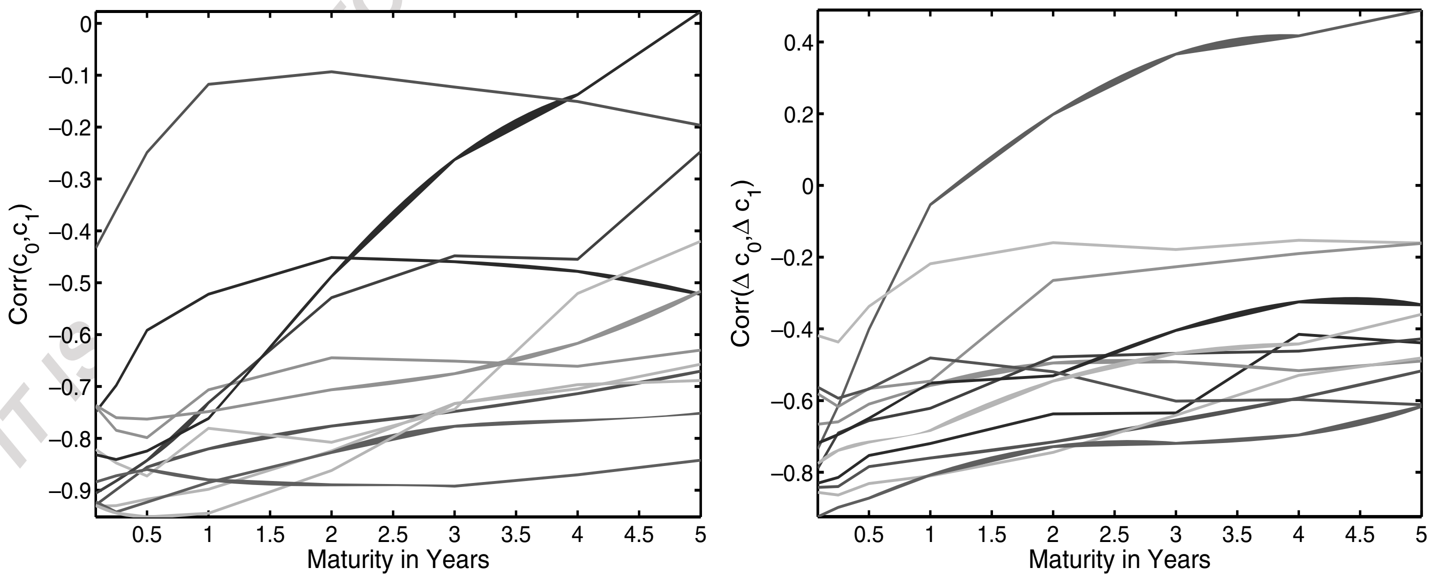
Foresi & Wu: Journal of Derivatives, Winter 2005

Negative Correlation Between Level and Slope

Correlation decreases with expiration

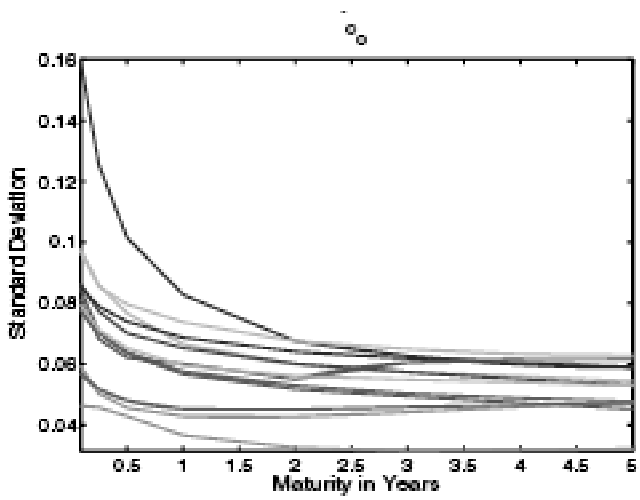
EXHIBIT 5

Cross Correlations between Volatility Level and Smirk Slope



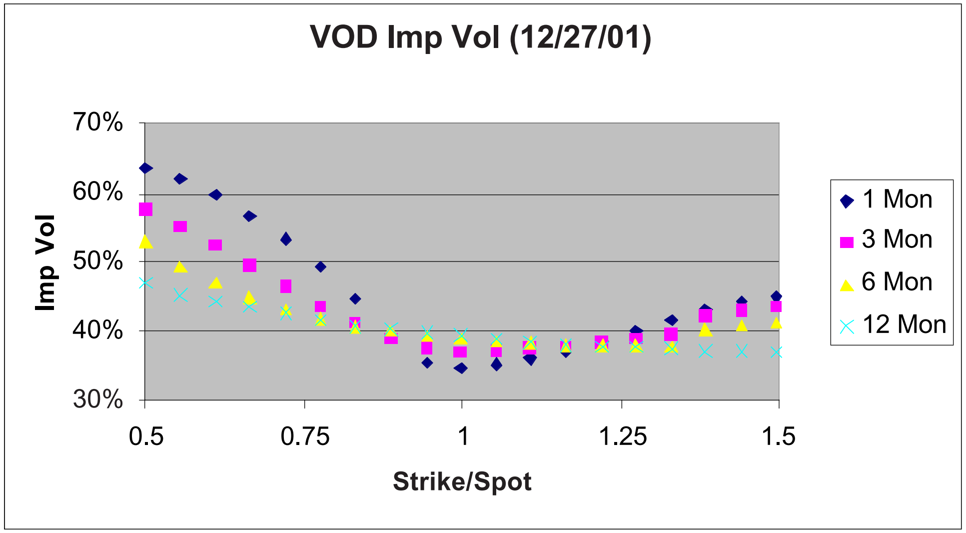
Lines denote the cross-correlation estimates between the volatility level proxy (c_0) and the volatility smirk slope proxy (c_1). The left panel measures the correlation based on daily estimates, the right panel measures the correlation based on daily changes of the estimates.

Volatility of Implied Volatility Decreases with Expiration

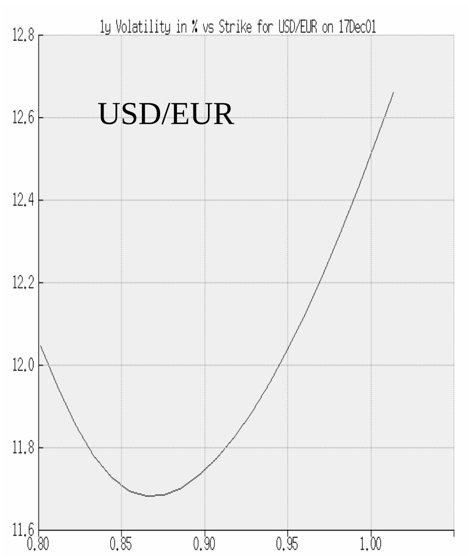


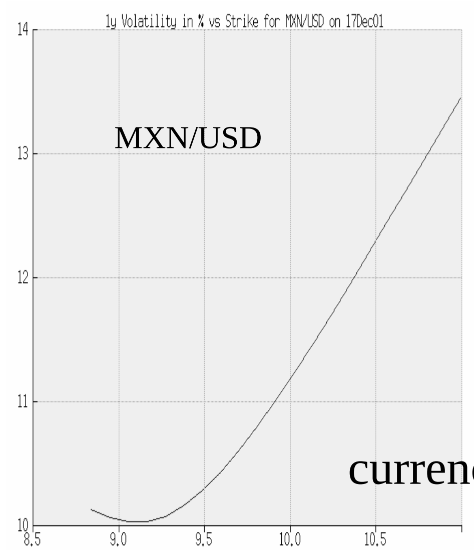
Short-term implied volatilities are more volatile. This suggests mean reversion or stationarity for the instantaneous volatility of the index.

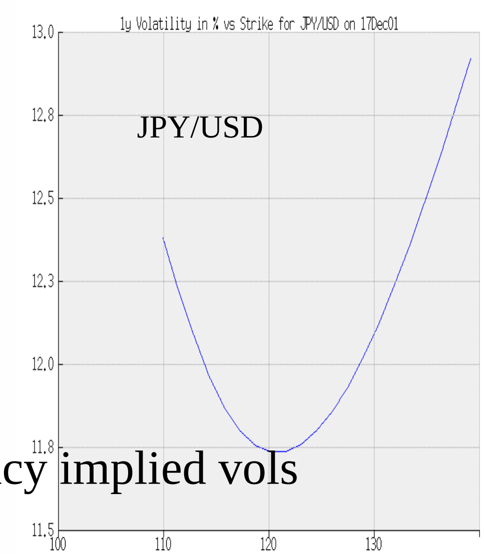
Other Markets



single stock implied vols







currency implied vols

ATM Strike = 0.90 9.85 123.67

The Smile is a Problem for Hedging

Black-Scholes is being used as a quoting mechanism for vanilla options (cf. yield to maturity for quoting bond prices) but is not the correct valuation/hedging model.

If the valuation formula is wrong, then the hedge ratio is wrong, therefore the price doesn't correspond to the correct replication formula. What's the right way to hedge index options?

Estimate of hedging error for an S&P index option with $S \sim 1000$ and $T = 1$ year

$$\Delta = \frac{dC_{BS}(S, t, K, T, \Sigma)}{dS} = \frac{\partial C_{BS}}{\partial S} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \qquad \text{Eq.1.1}$$

$$\frac{\partial C}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 400 \qquad \frac{\partial \Sigma}{\partial S} \sim \frac{\partial \Sigma}{\partial K} \sim \frac{0.02}{100} \sim 0.0002 \qquad \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} \sim 400 \times 0.0002 = 0.08$$

For a typical move $\delta S \approx 10$ S&P points the hedging error = 0.8 points

The P&L earned from perfect hedging is $\Gamma \times \frac{\delta S^2}{2} \sim \frac{1}{S\Sigma\sqrt{T}}\frac{\delta S^2}{2} \sim \frac{1}{200}(50) = 0.25$ points,

which is dwarfed by the error.

The Smile is a Problem for Valuing Exotics

Since Black-Scholes is inappropriate, we don't know how to value all sorts of exotic averages: barriers, averages, lookbacks … which effectively involve multiple "strikes" and therefore multiple implied volatilities.

Which one do you use in a Black-Scholes world?

How can we fix/replace/alter/correct Black-Scholes?

What Could Cause the Smile?

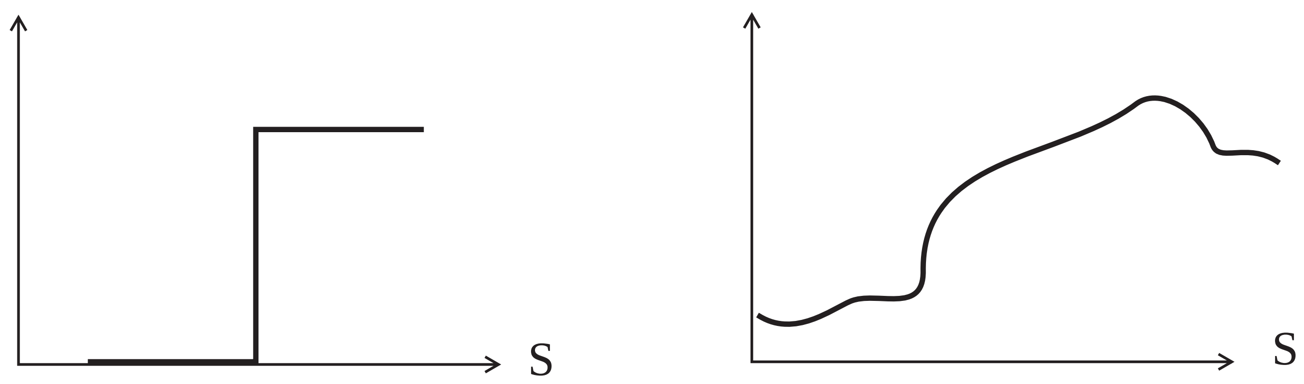
- Behavioral Causes
  - Crash protection/ Fear of crashes) [Katrina]
    Out-o- money puts rise relative to at-the-money; so do realized volatilities
  - Expectation of changes in volatility over time
  - Support/resistance levels at various strikes
  - Dealers' books tend to be filled by the demand for zero-cost collars
- Structural Causes: Violations of Black-Scholes assumptions
  - Inability to hedge continuously
  - Transactions costs
  - Local volatility $\sigma(S, t)$; leverage effect; CEV models
  - Stochastic volatility
  - Jumps/crashes

The Principles of Modeling

- Physics is about the future; finance is mostly about expectation of the future.
- Absolute valuation
  - Newton in physics; no such thing in finance
- Relative valuation: The only reliable law of finance
  - If you want to know the value of something (illiquid), find the market price of something similar and liquid.
  - Similar: same payoffs under *all* **(?)** circumstances.
  - static replication is best – needs no systems, virtually no assumptions.
  - dynamic replication is next best – but needs uncertain replication models, calibration, liquidity assumptions and a big investment in technology.
  - If you can't do either of those, then markets are incomplete and you need a utility function.
- A model is only a model: capture the important features, not all of them.

Static Replication of Exotic Options

- The best way to handle valuation in the presence of a smile is to use no model at all. For terminal payoffs:



- Any of these payoffs can be exactly duplicated by a linear combination of forwards, bonds, puts and calls with a range of strikes, and hence exactly replicated at a cost known if we know the volatility smile.
- Replication will hold irrespective of jumps, volatility, etc. -- only problem is counterparty risk.
- Try to do approximately the same for other exotic options.

Dynamic Replication: Local Volatility Models

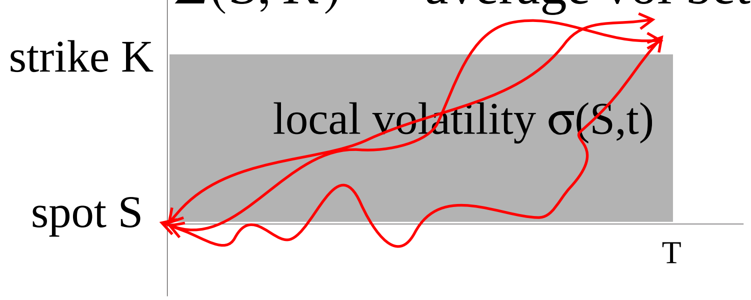
$$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ \qquad \text{Eq.1.2}$$

- $\sigma(S, t)$ is a *deterministic* function of a *stochastic variable* S. Can still replicate options, still do risk-neutral pricing.
- Calibration: $\sigma(S, t) \leftrightarrow \Sigma(S, t, K, T)$?
  Even if there is a solution, does the local volatility describe the asset?
- Why would local volatility make sense:
  - Leverage effect: as assets approach liabilities, equity volatility increases
  - Fear: increase in implied volatility as equity declines
  - CEV models: $dS = \mu(S, t)dt + \sigma S^{\beta} dZ$ but they are parametric and cannot match the implied volatility surface

The Smile in Local Volatility Models: First Look

- Interest rates: long rates are averages of expected future short rates between today and maturity
- In local volatility models
  implied volatilities are “averages” of expected future volatilities between spot and strike, between today and expiration.

$$\Sigma(S, K) \ = \ \text{average vol between S and K}$$



$$\Sigma(S, K) \approx \frac{\sigma(S) + \sigma(K)}{2}$$

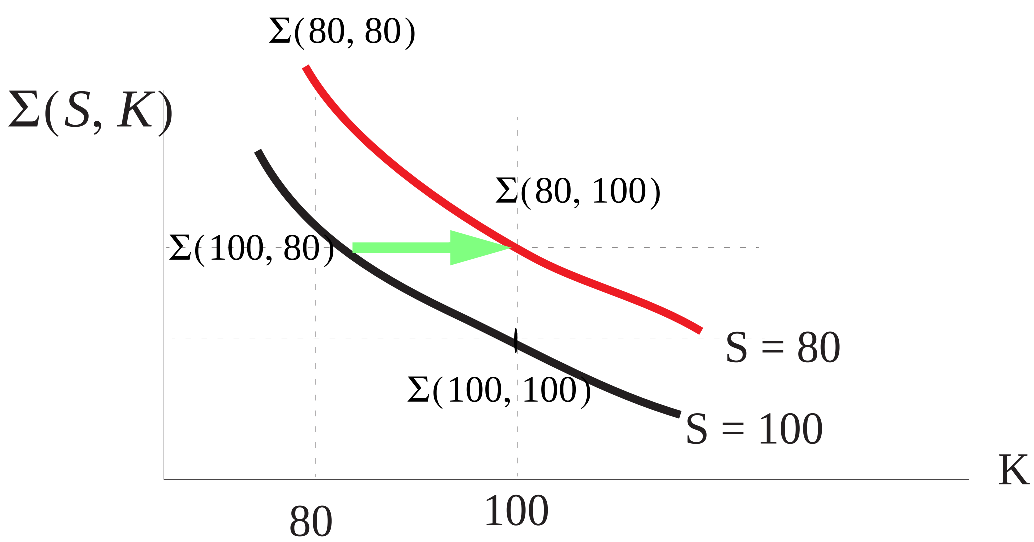
$$\left.\frac{\partial \Sigma}{\partial S}\right|_{S = K} \sim \left.\frac{\partial \Sigma}{\partial K}\right|_{S = K} \sim \frac{1}{2}\sigma(S)$$

Local volatilities predict \ the change of implied volatility with stock price, and that local volatility varies twice as fast with spot as implied volatility varies with strike.

Local Volatility Models ...

$$\Sigma \approx \Sigma(K+S)$$

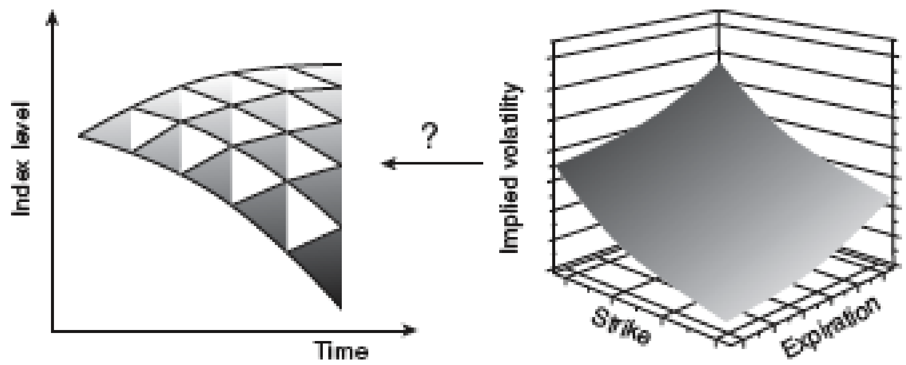
Let’s draw this for a roughly linear skew in the following figure.



Smile moves up when stock price moves down.

The Relation Between Local and Implied Vols

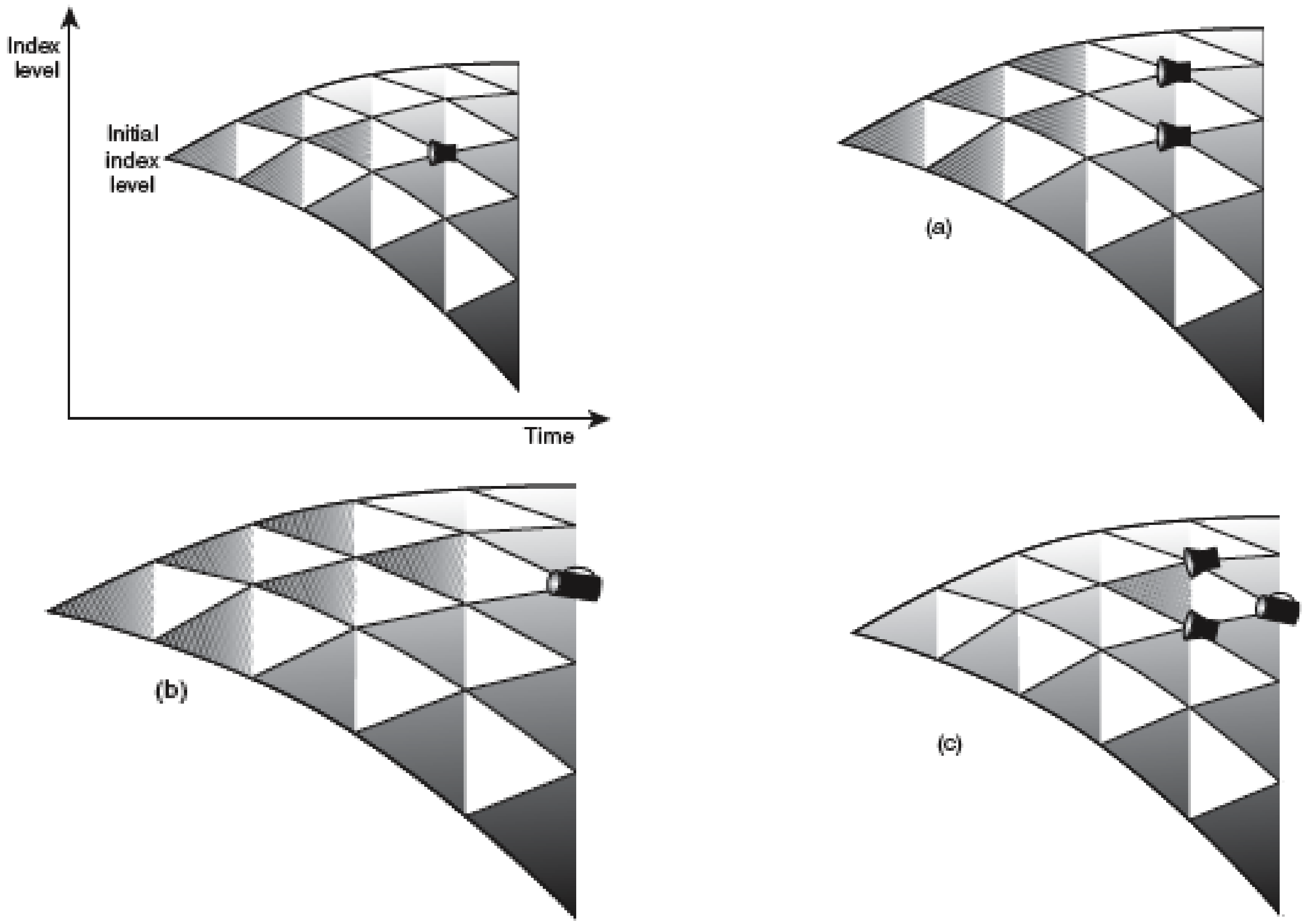
Think of implied volatilities as a complex probe/X-ray of the interior of the local volatility surface. Can one deduce local volatilities from implied?



This is an inverse scattering problem.

Inverse Scattering: Schematic Calibration

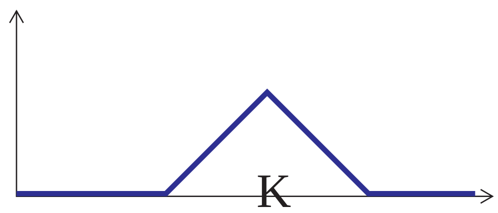
Derman-Kani binomial method



Dupire Continuous-Time Equation

Recall the Breeden-Litzenberger formula: risk-neutral probability/Arrow-Debreu price is a butterfly spread:

$$p(S, t, K, T) = e^{r(T-t)} \frac{\partial^2}{\partial K^2}(C(S, t, K, T))$$



We can go further than this, and actually find the local volatility $\sigma(S, t)$ from market prices of options. For zero interest rates and dividends, the local volatility at $S_T = K$ can be determined from the derivatives of option prices:

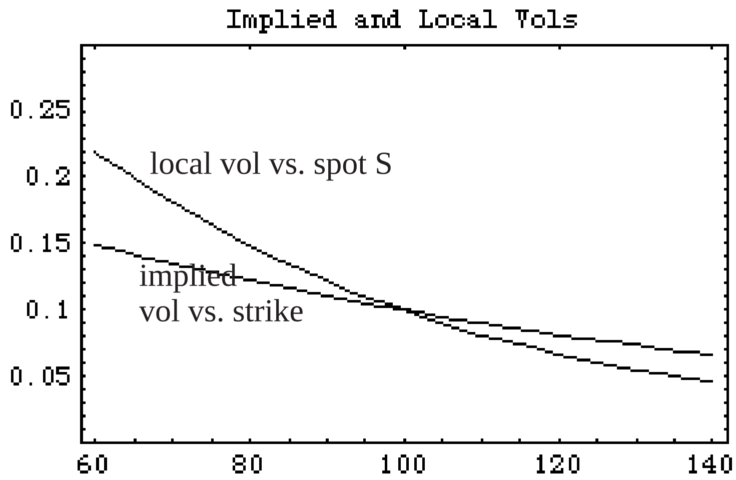
$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T} C(S, t, K, T)}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

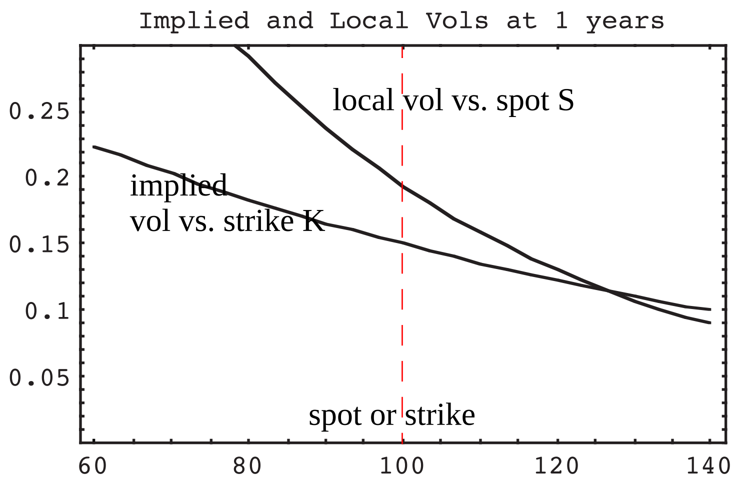
calendar spread ~ variance x prob

butterfly spread ~ variance

Example: Local Volatility Extraction

Local volatilities are the analogs of forward/sideways rates





Local Volatility ...

- In practice, we don't have a continuous implied volatility surface so we cannot use these relations blindly. Needs more sophisticated methods.
- Nice approximate harmonic average relation for short expirations due to Roger Lee:

$$x = \ln(S/K)$$

$$\frac{1}{\Sigma(x)} = \frac{1}{x}\int_{o}^{x} \frac{1}{\sigma(y)} dy$$

Stochastic Volatility Models

$$\begin{aligned} dS &= \mu_S(S, V, t)dt + \sigma_S(S, V, t)dZ_t \\ dV &= \mu_V(S, V, t)dt + \sigma_V(S, V, t)dW_t \\ V &= \sigma^2 \\ E[dWdZ] &= \rho dt \end{aligned}$$

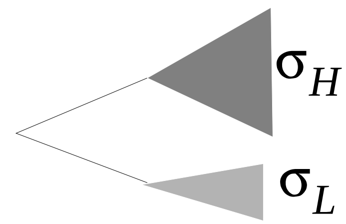
- Can replicate only if you can trade “volatility”/options as well as stock.
- Option prices are risk-neutral, “volatility” is not -- need market price of risk for volatility.
- You must hedge with stock and options - difficult!
- Can one really know the stochastic PDE for volatility? A “tall order.” (Rebonato).
- Correlation is even more stochastic than volatility.

The Smile in Stochastic Volatility Models

Poor man's point of view: Black-Scholes with the volatility in one of two regimes:

Take expectation of prices:

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)]$$



Because the Black-Scholes equation is homogeneous in S and K, we can write

$$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}(S, K, \Sigma)$$

So

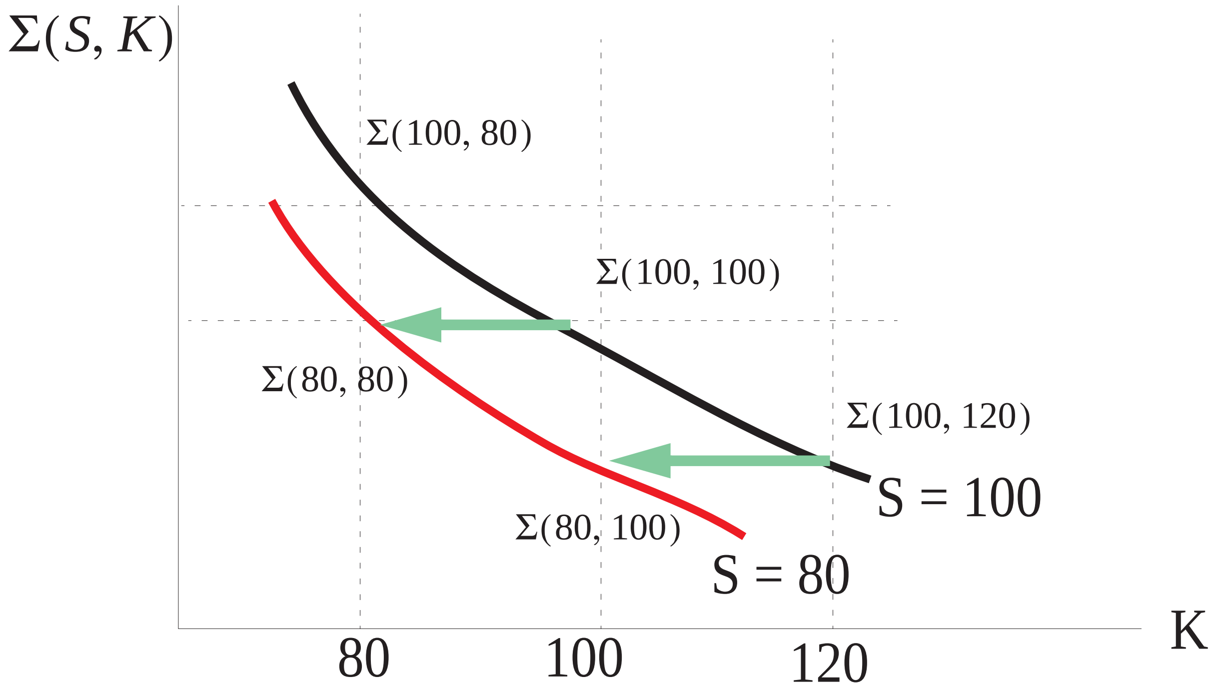
$$\Sigma = g\left(\frac{K}{S}\right) \qquad \frac{\partial \Sigma}{\partial S} \approx -\frac{\partial \Sigma}{\partial K} \text{ at the money}$$

Just the inverse of the relation in local volatility models.

The Smile in Stochastic Volatility Models …

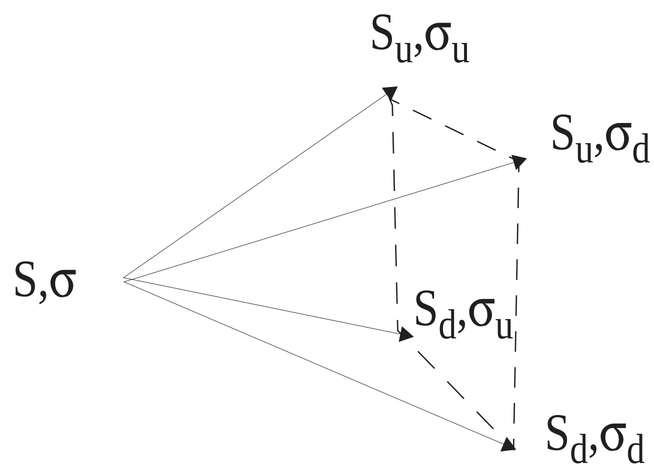
$$\Sigma \approx \Sigma(S - K)$$

Smile moves up when stock moves up, *if volatility doesn't change*.



Stochastic Volatility PDE

Volatility is mean-reverting: $d\sigma^2 = \alpha(m - \sigma^2)dt + \xi\sqrt{\sigma^2}\,dW$ r Heston, etc.



$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0$$

Risk neutrality: volatility drift or market price of risk ϕ is determined by options prices being risk-neutral.

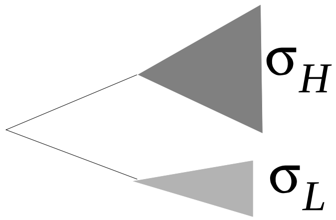
Stochastic Volatility Characteristic Solution

$$\overline{\sigma_T}^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

average variance along a path

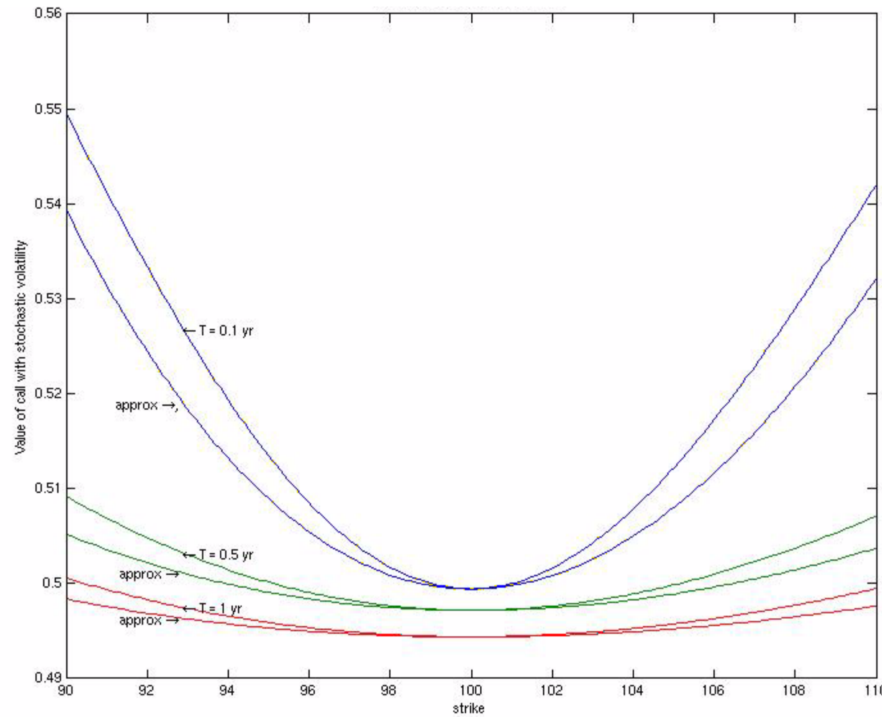
$$V = \sum_{\sigma_T} p(\sigma_T) \times BS(S, K, r, \sigma_T, T)$$

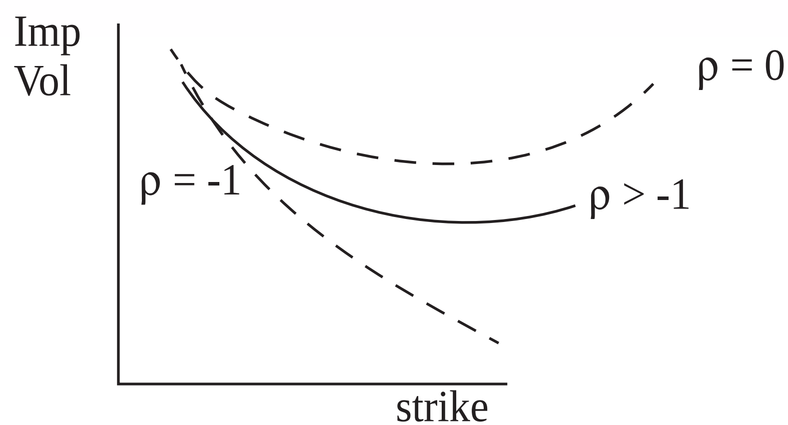
the Mixing Formula for $\rho = 0$



Stochastic Volatility Smiles:

- symmetric for $\rho = 0$, because of fat tails;
- highly curved for short expirations;
- flat for long expiration as mean reversion wins out.
- When $\rho \neq 0$ smile is asymmetric.
- You need very high volatility of volatility to get a steep short-term smile.
- If you cannot hedge with options, then the optimal pure stock hedge ratio looks a lot like the local volatility hedge ratio, i.e. $\Delta < \Delta_{BS}$ when $\rho < 0$
- Local volatility is the average of all stochastic volatilities at a definite future stock price and time.

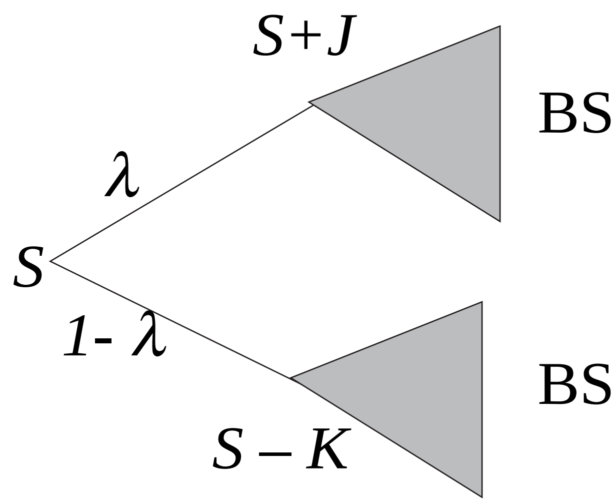




Jump-Diffusion Models

In fact, not only is volatility stochastic, but more seriously, stock prices move discontinuously. Geometric Brownian motion isn't the right description.

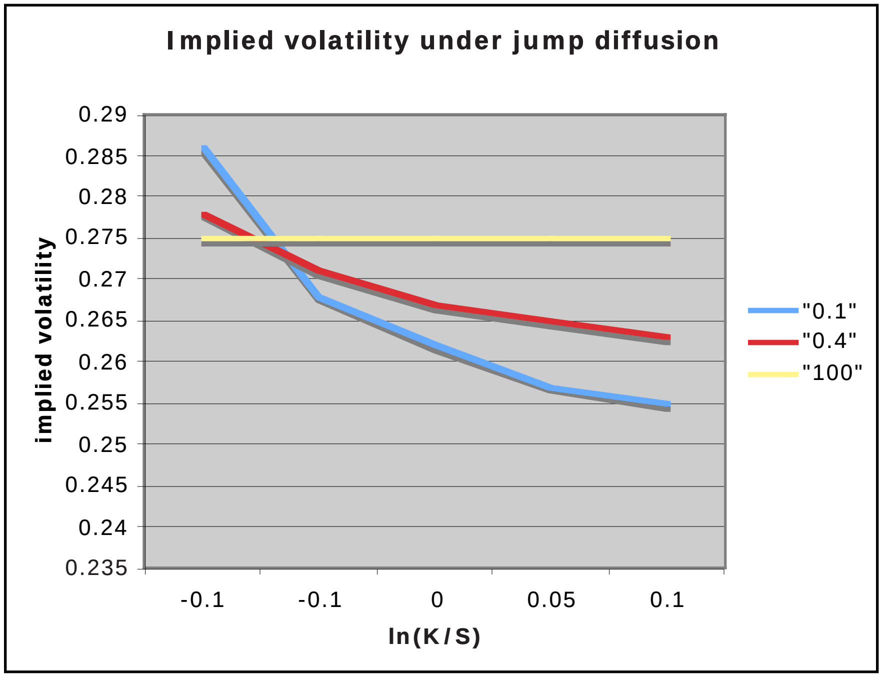
Merton: Poisson distribution of unhedgeable but diversifiable jumps.



$$C_{JD} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r + \frac{n\left(\bar{J} + \frac{1}{2}\sigma_J^2\right)}{\tau} - \lambda\left(e^{\bar{J} + \frac{1}{2}\sigma_J^2} - 1\right)\right)$$

Jump-Diffusion Smiles

- Steep realistic short-term smiles from the instantaneous jump.
- Long-term smiles tends to be flat as diffusion $\sigma^2\tau$ overwhelms the jumps.
- A higher jump frequency helps produce a steeper smile at expiration, because the jumps tends to happen more frequently and therefore are more likely to occur in the future as well.
- Stochastic volatility, in contrast, has difficulty producing a steep short-term smile because the volatility diffuses, and therefore doesn't change too much initially. The smiles in a stochastic volatility model are more pronounced at longer expirations.



Pragmatic Interpolation Methods

- Practitioners think of options models as interpolating formulas that take you from known prices of liquid securities to the unknown values of illiquid securities:
  - options;
  - convertible bonds.
- An exotic option is a mixture of ordinary options.
- Find a static hedge composed of a portfolio of vanilla options that approximately replicates the payoff or the vega of the exotic options.
- Then figure out the value of the portfolio of vanilla options in a skewed world.

Summary

- There is no “right” model.
- The best you can do is pick a model that mimics the most important behavior of the underlyer in your market. Then add perturbations if necessary.
- Examine the value of an option in a variety of plausible models.
- Try to be as model independent as possible.
- There is lots of work to be done on modeling realistic distributions and the options prices they imply.