CDO Modeling: Techniques, Examples and Applications

Christian Bluhm*

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Abstract.: Collateralized debt obligatons (CDOs) constitute an important subclass of asset backed securities. The evaluation of CDOs relies on mathematical modeling and on simulation as well as analytic and semi-analytic approaches, depending on the underlying asset pool and the cash flow structure of the transaction. This paper is an introductory survey on CDO modeling. It starts with a 'mini course' on the use of CDOs as capital market instruments, explains simulation and analytic approaches for evaluating CDOs and considers the notion of PD, EL and LGD of CDO tranches.

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Disclaimer. The content of this paper reflects the personal view of the author and not the opinion of HypoVereinsbank. The calculation and simulation examples in Sections 4 and 5 are based on realistic but fictitious transactions and data. Their sole purpose is to illustrate the concepts presented in this paper.

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*HypoVereinsbank (Munich), Group Credit Portfolio Management, GCP3 - Structured Finance Analytics. Comments are welcome and can be sent to the author by email: christian.bluhm@hvb.de

1 Introductory Remark

A main motivation for writing this survey paper on CDO modeling was an invitation to give the closing talk at the Autumn School on Risk Management (September 29 - October 02, 2003 in Herrsching/Ammersee) of the Munich University of Technology. The aim of the talk was to give a brief and self-contained introduction to CDOs as structured finance instruments from a mathematical modeling point of view. The slides of the talk (although not all of the topics discussed in this paper found their way onto them) can be downloaded at the Autumn School's website

www.mathematik.tu-muenchen.de/gkaam/AutumnSchool/index.html

The paper is divided in two parts. Sections 2 and 3 are written in prose and can be considered as a very brief and self-contained 'mini introduction' to CDOs as capital market instruments. Sections 4 and 5 present some ideas regarding the mathematics of CDOs. Readers only interested in the application of mathematical concepts to structured finance problems can safely omit Section 2. However, my hope is that readers without any experience in CDOs but interested in their functionality and applications will find Sections 2 and 3 useful.

A reference for readers interested in diving deeper into the mathematics of CDO modeling is [14], where a much more detailled and mathematically more rigorous course in CDO modeling will hopefully soon be 'ripe' for publication.

2 Four good reasons for CDO business

CDO issuance showed a remarkable growth during recent years. Although this growth has slowed down in 2003 due to a difficult economic environment, it can be expected that the growth of CDO issuance will continue over the next years.

What people not involved in CDOs typically ask is the following question: What makes this asset class so successful in the structured finance market, and why are people still considering new variations of the scheme with unbroken enthusiasm? It is the purpose of the next four sections to outline an answer to this question. Besides the four motivations summarized in the sequel, tax and legal arbitrage play an important role in CDO structuring. Because they are not described by mathematics, these aspects are not considered in this paper.

In the following we will look at two 'real life' examples and one 'illustrative' example. The first two examples refer to transactions done by members of HypoVereinsbank (HVB) Group in the past nine months. I have chosen these two particular structures from a variety of transactions done by HVB because they nicely illustrate two of the four main motivations for CDO issuance. Both transactions are public and the reader can find more details regarding the structures in the corresponding presale reports of the rating agencies. Obviously, our purpose here is not to discuss the performance or to disclose the economics of these transactions but rather to use them as examples for 'good reasons' why banks issue CDOs. Equivalently, one could have used purely illustrative examples based on fictitious transactions, as exercised in Sections 4 and 5. However, there is a good chance that readers will find it interesting to see at least some 'real' examples.

Section 2.1 is intended to be an introductory 'warm-up' and therefore more lengthy than the subsequent parts of this section.

2.1 Motivation 1: Spread arbitrage opportunities

We begin this section by quoting some passages from an article which appeared in the web at www.FinanceAsia.com (written by Rob Davies, March 20, 2003):

HVB Asset Management Asia (HVBAM) has brought to market the first ever hybrid collateralized debt obligation (CDO) managed by an Asian collateral manager. The deal, on which HVB Asia (formerly known as HypoVereinsbank Asia) acted as lead manager and underwriter, is backed by 120 *million of asset-backed securitization bonds and* 880 *million of credit default swaps ... Under the structure of the transaction, Artemus Strategic Asian Credit Fund Limited - a special purpose vehicle registered in the Cayman Islands - issued* 200 *million of bonds to purchase the* 120 *million of cash bonds and deposit* 80 *million into the guaranteed investment contract, provided by AIG Financial Products. In addition, the issuer enters into credit default swap agreements with three counterparties (BNP Paribas, Deutsche Bank and JPMorgan) with a notional value of* 880 *million. On each interest payment date, the issuer, after payments of certain senior fees and expenses and the super senior swap premium, will use the remaining interest collections from the GIC accounts, the cash ABS bonds, the hedge agreements, and the CDS premiums from the CDS to pay investors in the CDO transaction ... The transaction was split into five tranches, including an unrated* 20 *million junior piece to be retained by HVBAM. The* 127 *million of A-class notes have triple-A ratings from Fitch, Moody's and S&P, the* 20 *million B-notes were rated AA/Aa2/AA, the* 20 *million C bonds were rated A/A2/A, while the* 13 *million of D notes have ratings of BBB/Baa2 and BBB.*

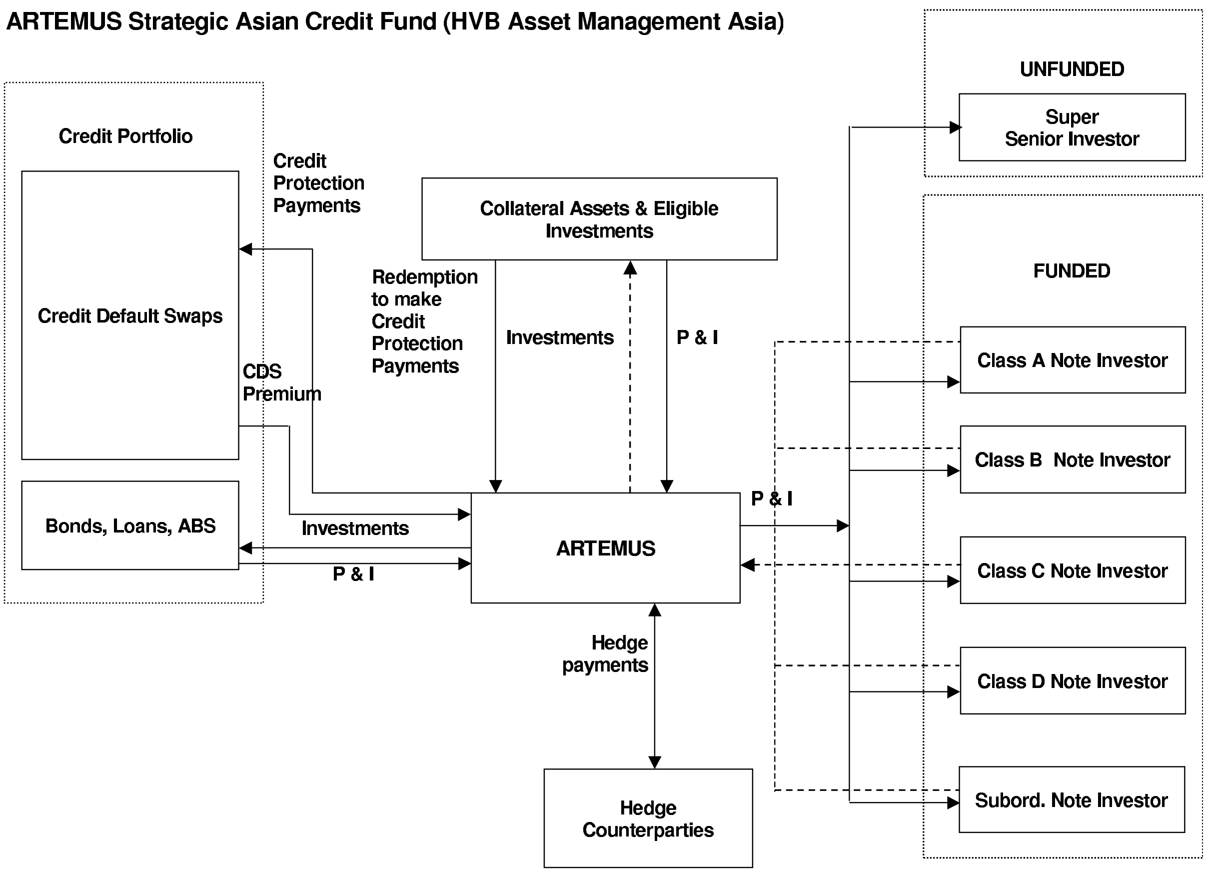

Figure 1: Artemus Strategic Asian Credit Fund

Figure 1 shows a structural diagram of the transaction. As mentioned in the quotation from FinanceAsia above, the three major rating agencies analyzed the transaction. The reader can find the results of their analysis in the three presale reports [1], [2] and [3] of the transaction.

We now want to look at the transaction in a more detailled manner. Hereby we focus on some aspects only and try to simplify things as much as possible. For example, hedge agreements on interest rates and currencies will be excluded from our discussion.

In the sequel, all amounts of money refer to USD.

Liability side of the structure

The issuer (Artemus Strategic Asian Credit Fund Limited, an SPV at Cayman, from now on shortly called 'Artemus') issued 200mm of bonds, split in five tranches reflecting different risk-return profiles. Artemus (as protection buyer) also entered into a CDS agreement (super senior swap) on a notional amount of 800mm with a super senior swap counterparty. Such counterparties (protection sellers on super senior swaps) are typically OECD-banks with excellent credit quality. Because the liability side has a funded (200mm of notes) and an unfunded (800mm super senior swap) part, the transaction is called partially funded.

Asset side of the structure

The proceeds of the 200mm issuance have been invested in a guaranteed investment contract (GIC account; 80mm in eligible collateral assets) and asset backed securities (ABS bonds; 120mm). Additionally, the issuer sold protection on a pool of names with an aggregated notional amount of 880mm. Because the asset side consists of a mixture of debt securities and synthetic assets (CDS), the transaction is called hybrid. Note that the GIC is kind of 'risk-free' (AAA-rated securities, cash-equivalent).

Settlement of credit events

If credit events happen on the 880mm CDS agreement (remember: Artemus is protection seller), a settlement waterfall takes place as follows.

- Proceeds from the GIC account are used by Artemus to make payments on the CDS agreement.
- If proceeds from the GIC are not sufficient to cover losses, principal proceeds from the debt securities are used to pay for losses.
- If losses exceed the notional amount of the GIC and principal proceeds, then ABS securities are liquidated and proceeds from such liquidation are used for payments on the 880mm CDS agreement.
- Only if all of the above mentioned funds are not sufficient for covering losses, the super senior swap will be drawn (remember: Artemus bought protection from the super senior swap counterparty).

Note that (at start) the volume of the GIC plus the super senior swap notional amount exactly match the 880mm CDS agreement, and that the 120mm ABS Securities plus the 880mm CDS volume 'asset-back' the 1bn total tranche volume on the liability side. However, these coverage equations refer only to principal and swap notionals outstanding. But there is much more credit enhancement in the structure, because additional to the settlement waterfall, interest proceeds, mainly coming from the premium payments on the 880mm CDS agreement and from the ABS bonds, mitigate losses as explained in the following section.

Distribution of proceeds

Principal proceeds (repayment/amortization of debt securities) and interest proceeds (income on ABS bonds, the GIC, hedge agreements and premium from the 880mm CDS agreement) are generally distributed sequentially top-down to the note holders in the order of their seniority. On top of the interest waterfall, fees, hedge costs and other senior expenses and the super senior swap premium have to be paid. Both, principal and interest payments are subject to change in case certain coverage test are broken. There are typically two types of coverage tests in such structures:

- Overcollateralization tests (O/C) take care that the available (principal) funds in the structure are sufficient for a certain (over)coverage (encoded by O/C-ratios greater than 100%) regarding repayments due on the liability side of the transaction.

- Interest coverage tests (I/C) make sure that any expenses and interest payments due on the liability side of the structure and due to other counterparties involved, e.g., hedge counterparties, are (over)covered (encoded by I/C-ratios greater than 100%) by the remaining (interest) funds of the transaction.

If a test is broken, cash typically is redirected in a way trying to bring the broken test in line again. In this way, the interest stream is used to mitigate losses by means of a changed waterfall. It is beyond the scope of this paper to dive deeper into such cash flow mechanisms.

Excess spread

As already mentioned above, interest proceeds are distributed top-down to the note holders of classes A, B, C and D. All excess cash left-over after senior payments and payments of coupons on classes A to D is paid to the subordinated note investors. Here, HVB Asset Management Asia (HVBAM) retained part of the subordinated note (the so-called equity piece). Such a constellation is typical in arbitrage structures: Most often, the originator/arranger keeps some part of the most junior piece in order to participate in the excess spread of the interest waterfall. Additionally, retaining part of the ‘first loss’ of a CDO to some extent ‘proves’ to the market that the originator/arranger itself trusts in the structure and the underlying credits. As indicated above in our discussion on coverage tests, if tests are broken excess cash typically is redirected in order to protect senior note holder’s interests. Here, the timing of defaults (see Section 5) is essential: If defaults occur at the end of the lifetime of the deal (backloaded), subordinated notes investors had plenty of time to collect excess spread and typically will achieve an attractive overall return on their investment even if they loose a substantial part of their invested capital. In contrast, if defaults occur at an early stage of the transaction (frontloaded), excess cash will be redirected and no longer distributed to the equity investor. This is a bad scenario for equity investors, because they bear the first loss (will loose money) but now additionally miss their (spread) upside potential because excess cash is trapped.

Where does the arbitrage come from?

Now where does the arbitrage come from? The key observation is that on the 880mm CDS agreement and on the 120mm ABS securities on the asset side premiums are collected on a *single-name* base, whereas premium/interest payments to the super senior swap counterparty and the note holders refer to a *diversified pool* of ABS bonds and CDS names. Additionally, the tranching of the liability side into risk classes contributes to the spread arbitrage in that tranches can be sold for a comparably low spread if sufficient credit enhancement (e.g., subordinated capital, excess cash trapping, etc.) is built-up for the protection of senior notes.

Obviously, the total spread collected on single credit risky instruments at the asset side of the transaction exceeds the total 'diversified' spread to be paid to investors on the tranched liability side of the structure. Such a mismatch typically creates a significant arbitrage potential which offers an attractive excess spread to the equity or subordinated notes investor.

There are many such transactions motivated by spread arbitrage opportunities in the CDO market. In some cases, structures involve a so-called rating arbitrage which arises whenever spreads increase quickly and rapidly and the corresponding ratings do not react fast enough to reflect the increased risk of the instruments. Rating arbitrage as a phenomenon is an important reason why typically a serious analysis of arbitrage CDOs should not rely on ratings alone but also considers all kinds of other sources of information.

From a modeling perspective, arbitrage structures constitute an interesting and challenging class of CDOs because in most cases all kinds of cash flow elements are involved.

Looking at arbitrage structures from an economic point of view one could say that a well-structured transaction like Artemus Strategic Asian Credit Fund has - due to the arbitrage spread involved - *a potential to offer an interesting risk-return profile to notes investors as well as to the originator/arranger holding (part of) an unrated junior piece.* It is certainly possible that the incorporated spread arbitrage is sufficiently high to compensate both groups of people adequately for the risk taken.

A word on super senior swaps

Regarding super senior swaps one should mention that in most transactions the likelihood that the super senior tranche gets hit by a loss will be close to zero. *Scenarios hitting such a tranche typically are located far out in the tail of the loss distribution of the underlying reference pool.* Looking at super senior swaps from a heuristic (non-mathematical) point of view, one can say that in order to cause a hit on a super senior tranche the economy has to turn down so heavily that it is very likely that problems will have reached a level where a super senior swap hit is just the tip of the iceberg of a heavy global financial crisis.

2.2 Motivation 2: Regulatory capital relief

Regulatory capital relief is another major motivation why banks issue CDOs. Here, most often the 'D' in 'CDO' becomes an 'L' standing for 'loan'. The CDO is then called a collateralized loan obligation (CLO). Figure 2 shows a typical CLO issued for the purpose of regulatory capital relief. Again, the reader has the option to look into presale reports of the rating agencies; see [4] and [5]. In Building Comfort 2002-1, HVB has bought protection on a pool of 5bn Euro of residential mortgage backed loans. In this particular transaction, not even an SPV has been set-up. Instead, HVB directly issued notes on the lower 5% of the notional volume (including a 50bps equity piece

BUILDING COMFORT 2002 (HVB AG)

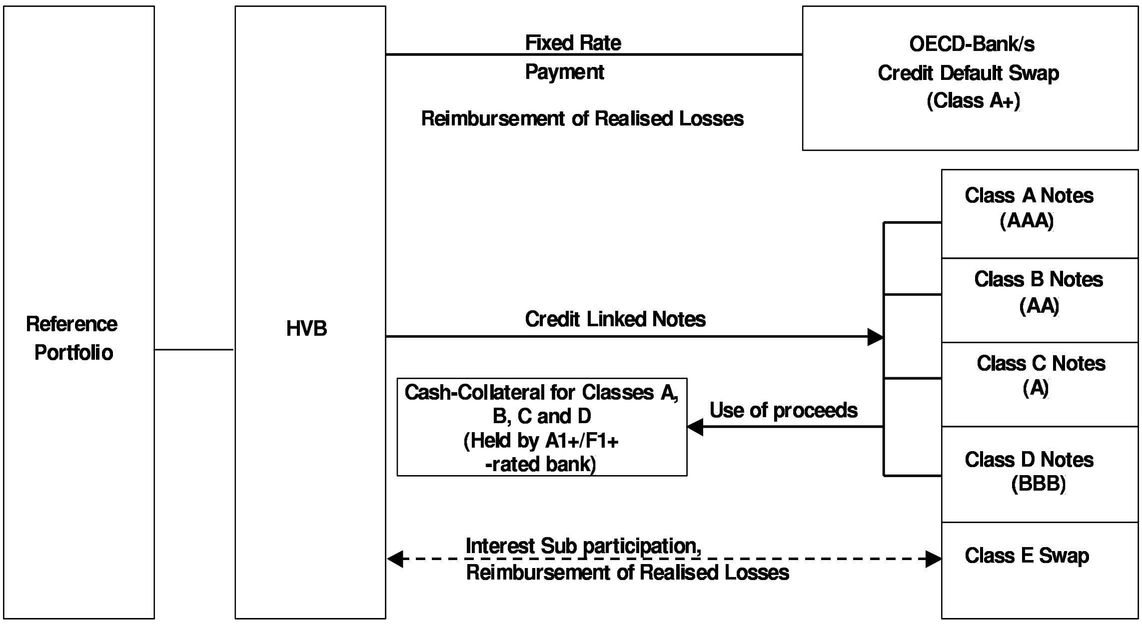

Figure 2: Building Comfort 2002-1

in form of a swap, the so-called Class-E Swap) and bought protection from an OECD-bank on the upper (senior) 95% of the notional volume of the RMBS pool.

Let us briefly outline what such a transaction means for the regulatory capital requirement of the underlying reference pool. In general, loan pools require regulatory capital in size of $8\%\times$RWA where RWA denotes the risk-weighted assets of the reference pool. Ignoring collateral eligible for a risk weight reduction, regulatory capital equals 8% of the pool's notional amount. After the (synthetic) securitization of the pool, the only regulatory capital requirement the originator has to fulfill regarding the securitized loan pool is holding capital for retained pieces. For example, if HVB would hold the equity piece, the regulatory capital required on the pool would have been reduced by securitization from 8% down to 50bps, which is the size of the equity tranche. The 50bps come from the fact that retained equity pieces typically require a full capital deduction. As already mentioned, in the special case of Building Comfort, HVB has implemented a swap (the Class-E Swap, with a so-called interest subparticipation) into the equity piece.

Altogether one clearly sees that structures like Building Comfort are very useful tools if one is interested in relief of regulatory capital. Even if a bank keeps an equity piece of, say, 3%, the securitization implies a 5% relief of regulatory capital, ignoring collateral eligible for RWA reduction again. As 'opportunity costs' for capital relief, the originating bank has to pay interest to notes investors, a super senior swap premium, upfront costs (rating agencies, lawyers, structuring and underwriting costs) ongoing administration costs and possibly some other expenses. A full calculation of costs compared to the decline of regulatory capital costs is required to judge about the economics of such transactions.

2.3 Motivation 3: Funding

There is not much to say about this third motivation banks have for CDO issuance. In so-called true sale transactions, the transfer of assets is not made of derivative constructions but rather involves a true sale 'off balance sheet' of the underlying reference assets to an SPV which then issues notes in order to refinance/fund the assets purchased from the originating bank. The advantage for the originator is the receipt of cash (funding).

Funding can be an issue for banks whose rating has declined to a level where funding from other sources is expensive. The advantage of refinancing by means of securitizations is that resulting funding costs are mainly related to the credit quality of the transferred assets and not so much to the rating of the originator. However, there remains some linkage to the originator's rating, if the SPV also enters into a servicer agreement with the originating bank. In such cases, investors and rating agencies will evaluate the servicer risk inherent in the transaction.

2.4 Motivation 4: Economic risk transfer

The last of the four major motivations for CDO issuance is economic risk transfer. Figure 3 illustrates the impact of a securitization transaction on the loss distribution of the underlying reference pool. Such a transaction divides the loss distribution in two segments. The left segment, the so-called first loss, refers to losses carried by the originator (e.g., by retaining the equity piece). The right segment refers to the excess loss of the first loss piece, taken by the CDO investors. The upper boundary of the first loss piece is an effective loss cap the originator 'buys' from the CDO investors. The chart on the right in Figure 3 illustrates the lucky situation that the securitization costs are much lower than the decline of the expected loss. In such situations, the risk-return profile of the securitized pool will be improved by the securitization transaction.

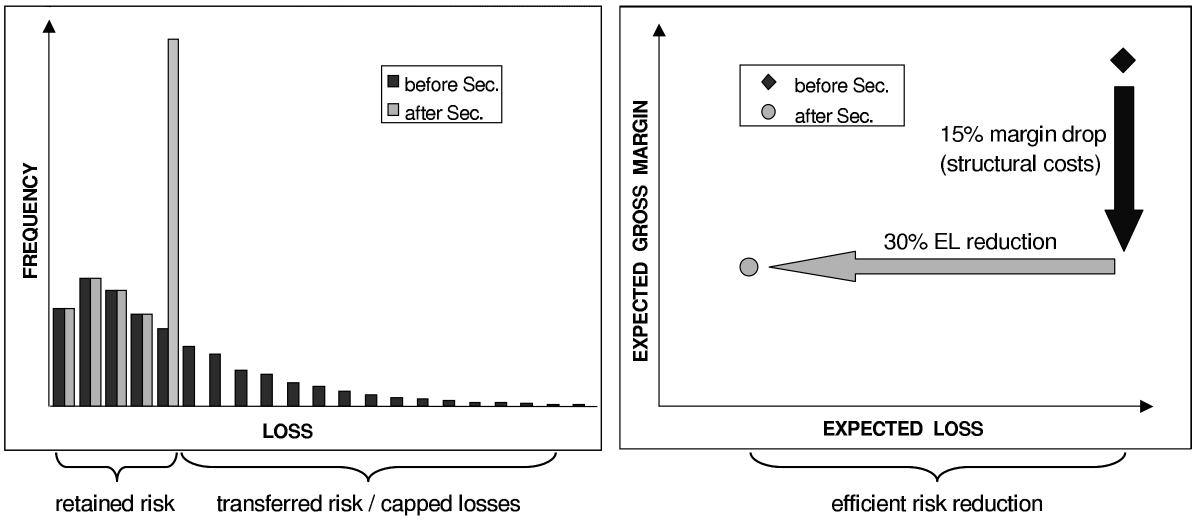

Figure 3: Economic risk transfer (illustrative!)

The insurance paradigm

A standard question people ask regarding risk transfer in the context of securitizations is the following: How can these deals transfer risk, if the first loss (often significantly higher than the expected loss of the original pool) is kept by the originator? To answer this question, we go back to the insurance paradigm sometimes used to explain the necessity of taking into account expected and unexpected losses in credit pricing. In this paradigm the expected loss is calculated as an insurance premium to be charged and deposited in some loss reserve account in order to cover the historic mean losses observed w.r.t. the considered class of credit risky instrument. Analogously, a capital cushion against unexpected losses is calculated by means of an economic capital definition (e.g., credit VaR, expected shortfall or some other risk measure). Now, if a bank securitizes a credit portfolio and retains only the first loss piece (FLP), there is some risk transfer if and only if the probability $\mathbb{P}[\text{Loss} > \text{FLP}]$ is greater than zero. Applying the insurance paradigm to this case, the insurance premium for covering losses can be chosen somewhat lower after securitizing the pool because there is an effective loss cap in place for the benefit of the originating bank.

Costs versus benefit

Summarizing one can say that most securitization transactions actually lead to some risk transfer. The problem is that the risk cost saving alone does not always justify the securitization costs spent on the liability side of a CLO (spread payments, hedge costs and other fees and costs). Moreover, the upfront costs of setting up a CLO can be quite expensive too. There will be expenses for structuring, underwriting, rating agency fees, lawyer costs, etc. Such costs hit the P&L of an originating bank right at the start of the transaction.

Impact on the contributory economic capital

Another important point for investigation in the context of risk transfer is the change in contributory economic capital implied by a securitization. The problem is that securitizing a subportfolio can cause some negative effect on the economic capital of the residual source portfolio due to the diversification turn-down caused by taking away a pool of diversifying assets; see [13], pages 256-258. Obviously, if the volume of the securitized pool is ‘small’ compared to the volume of the source portfolio, such negative impact has a chance to be negligible.

Motivation for Monte Carlo simulation tools

Besides others, all of the above mentioned aspects have to be taken into account by originating banks. Without tailor-made mathematical tools for evaluating the planned CLO in a way consistent with the bank’s internal portfolio model it is impossible to draw a complete picture of the impact of a securitization transaction. Of course, for CDO investors the same principles hold.

3 CDOs from a quantitative perspective

In this section we briefly indicate how CDO evaluation is done in general by means of Monte Carlo simulations. The general picture we have in mind is given in Figure 4.

On the left side of a CDO there will be always some pool of credit risky instruments, e.g., loans, bonds, credit derivatives (e.g., CDS), ABS notes, or even a combination of different asset classes. We refer (and referred in Section 2) to this part of a CDO as the asset side of the structure. On the right side of a CDO we typically have securities issued in the capital market. For obvious reasons we call this part of the CDO the liability side of the transaction. The two sides of the CDO are connected through the structural definition of the transaction. Typically the structure is represented

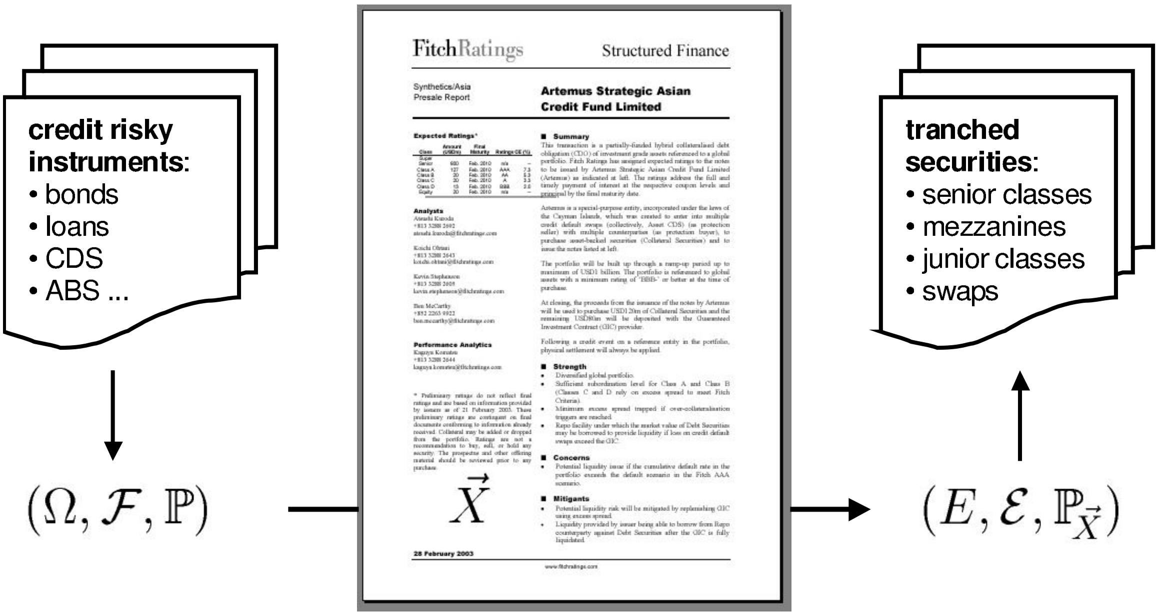

Figure 4: Modeling CDOs - cause and response

by means of a term sheet, an indenture, an offering circular or whatever documentation the investor receives from the originator/arranger.

Now, the key to a basic understanding of CDO modeling is the trivial fact that the only[1] reason for uncertainty regarding the performance of the securities on the liability side of the structure is the uncertainty regarding the performance of the underlying asset pool. In other words, *if we would be able to predict the economic future of the underlying assets with certainty, we would - just by applying the structural definitions (cash flows, etc.) - also be able to exactly predict the performance of securities on the liability side of the transaction.*

Mathematically speaking (see Figure 4), the asset side induces a probability space capturing the randomness of the underlying assets. The cash flow structure of the transaction uniquely defines a mapping $\vec{X}$ on this probability space, transforming asset scenarios into liability scenarios. A liability scenario can be thought of as a vector whose components include all kinds of numbers relevant for describing the performance of issued securities, e.g., tranche losses, coupons, IRRs, etc.; see [13], Section 8.3.

3.1 Modeling the asset side of a CDO

The probability space underlying the asset side of a transaction can be constructed by means of a credit portfolio model. There are various ways to implement a credit risk model suitable for CDO evaluation. In this paper, we restrict ourselves to a 'correlated default times' model; see Section 4 for an introduction to the mathematics of default times.

The basic idea of how to evaluate a CDO by means of default times is as follows. An asset scenario in a default times model basically consists of a vector $(\tau_1, ..., \tau_m)$ of default times for a portfolio of, say, m obligors in the asset pool underlying the CDO. The randomly drawn numbers τ_i represent

[1] If the asset pool is *managed*, the performance of the asset manager is another source of uncertainty.

the time until obligor i defaults. Based on an asset scenario, cash flows can be transformed into default-times-adjusted cash flows. Figure 5 illustrates the cash flow transformation by means of a bond. Until the default time τ which depends on the credit quality of the obligor, all cash flows are in place as scheduled. But at the time of default, cash flows dry-out and the bond investor receives some recovery amount. Note that in reality most often there will be some delay until the final settlement/work-out of the defaulted asset. In derivative transactions the time until settlement most often follows an ISDA master agreement.

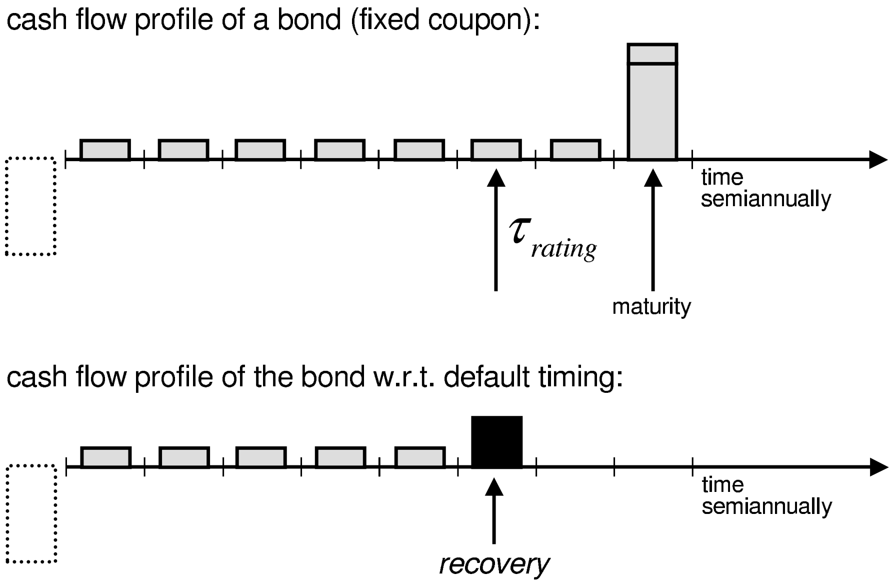

Figure 5: Transformation of cash flows by default times

In this way, cash flow CDOs, where the cash collected on the underlying assets is distributed on the liability side of the structure, can be evaluated. For purely synthetic structures the application of default times is even more straightforward, because it is primarily loss and default distributions that matter. More details on default times follow later.

3.2 Modeling the liability side of a CDO

The modeling of the structure of a CDO itself is far from being trivial. First of all it involves a careful study of the available documentation, starting from presales and ending at a voluminous offering memorandum. It also involves a careful balance between modeling the 'full range' of all cash flow elements and simplifying structural elements for the benefit of a better handling. Taking shortcuts regarding cash flow elements can be dangerous, of course, and has to be done with great care. To illustrate this, we give two examples.

Example 1: A harmless simplification

Assume that a CDO-SPV issues tranches where one tranche is split and issued in two currencies, e.g., 40% of the tranche are Euro denominated (Euro LIBOR as reference interest rate) and 60% of the tranche are Sterling denominated (Sterling LIBOR referenced). Lets say the underlying assets are all Sterling denominated and (Sterling LIBOR) floating rate notes. Obviously there is some currency mismatch inherent in the CDO which typically is hedged by means of a basis swap and a currency swap. The good news regarding the CDO model is that as long as the hedges are in place, there is no need to model the randomness underlying the currency risk. Instead, the hedge costs can just be implemented as another (senior) deduction from available funds in the interest waterfall of the CDO.

Example 2: A dangerous simplification

A less harmless shortcut is the following situation. An investor considers buying a mezzanine tranche of a cash flow CDO. In order to come to a quick decision, the investor only ‘tranches-up’ the loss distribution of the underlying pool (see 4.4) in order to get an estimate of the mezzanine tranche’s default probability and expected loss. Now, if the CDO is not just a ‘plain vanilla’ structure but incorporates some redirection of cash flows based on credit events in the asset pool (which will be the case in almost all cases of cash flow deals), such a ‘tranched loss’ approach (ignoring cash flow triggers and waterfall changes) is very likely to be quite misleading. For more sophisticated structures at least a semi-analytic approach (see 4.5), or even better, a ‘full’ Monte Carlo simulation approach is recommended.

4 CDO modeling techniques

This section will outline the basics of correlated default times. As a reference to research papers following more or less the same route we mention FINGER [20], LI [26, 27] and SCHMIDT and WARD [33].

Because default times depend on the credit quality of the considered obligor, well-calibrated credit curves are a main ‘ingredient’ for constructing default times. The derivation of such curves is the topic of the following section.

4.1 Calibration of a credit curve

For reasons of simplicity and to make the applicability clear, we asssume in this section that we have a set of ratings $R =$AAA, AA, A, BBB, BB, B, CCC, D, where AAA as always denotes the best credit quality, CCC refers to the worst non-defaulted credit quality and D denotes the default state. It is straightforward to find generalizations of the following results for finer rating scales.

It is best practice to assign a unique default probability (PD, in Basel II notation) to every obligor rating R. Table 1 shows an example of such a PD-calibration of ratings.

Rating	1-year PD
AAA	0.01%
AA	0.02%
A	0.08%
BBB	0.36%
BB	1.55%
B	6.75%
CCC	29.35%

Table 1: One-year PDs calibrated to ratings

The aim of this section is to calibrate a credit curve for each of the 7 ratings R, where a credit curve for a rating R is a mapping

$$t \mapsto p_t^{(R)} \ = \ \mathbb{P}[R \rightharpoonup \text{D in time } t] \qquad (t \geq 0;\ R \in \{AAA, AA, ..., CCC\}),$$

where '$\rightharpoonup$' denotes migration. In other words, $p_t^{(R)}$ is the probability that an obligor with a current rating of R defaults within the next t years. An example for probabilities

$$p_t^{(AAA)},\ p_t^{(AA)},\ \ldots, p_t^{(CCC)}$$

for $t = 1$ is given in Table 1. We will always count time in years.

There are many ways to do this. Here, we follow a well-known Markov chain approach (see JARROW ET AL. [24], ISRAEL ET AL. [23], KREININ and SIDELNIKOVA [25]), based on a one-year migration matrix from Standard & Poor's ([36], Table 8, Page 13). In practice, the calibration of credit curves is not as straightforward as exercised here. For example, internal information on credit migrations and default history will be a main source of data taken into account for the calibration of credit curves. However, our purpose here is to demonstrate how such a calibration can be done in principle. For our little exercise, we rely on rating agency data.

Rating agencies annually publish discrete credit curves $(p_t^{(R)})_{t=1,2,3...}$ based on cohorts of historically observed default frequencies; [36] is an example. The problem with historically observed cohortes is that the resulting multi-year PDs have a tendency to look kind of 'saturated' at longer horizons due to lack-of-data problems. Comparing corporate bond default reports from 5 years ago with reports as of today, one certainly recognizes a lot of improvement on the data side. Curves are smoother and do not imply zero forward PDs too early. Nevertheless the data quality is still not satisfactory enough to rely on historic data without 'smoothing' by some suitable model. The following Markov chain approach is an elegant way to overcome this problem and to generate continuous-time credit curves.

Let us now start with the adjusted average one-year migration matrix from S&P (see [36], Table 8, Page 13). We overcome the zero default observation problem for AAA-ratings by replacing the default column by the values from Table 1. In fact, the PDs in Table 1 have been calibrated based on a linear regression on a log-scale of the original default column in the S&P-matrix. To assure that we have a stochastic (migration) matrix, i.e., row sums equal to 1, we renormalize the rows of the modified S&P-matrix. As a result we obtain the one-year migration matrix $M = (m_{ij})_{i,j=1,...,8}$ presented in Table 2. Next, we need the following theorem.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	93.06%	6.29%	0.45%	0.14%	0.06%	0.00%	0.00%	0.01%
AA	0.59%	91.00%	7.59%	0.61%	0.06%	0.11%	0.02%	0.02%
A	0.05%	2.11%	91.43%	5.63%	0.47%	0.19%	0.04%	0.08%
BBB	0.03%	0.23%	4.44%	89.01%	4.70%	0.95%	0.28%	0.36%
BB	0.04%	0.09%	0.44%	6.07%	82.70%	7.89%	1.22%	1.55%
B	0.00%	0.08%	0.29%	0.41%	5.33%	82.23%	4.91%	6.75%
CCC	0.10%	0.00%	0.32%	0.65%	1.62%	10.30%	57.65%	29.35%
D	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%

Table 2: Modified S&P average one-year migration matrix

Theorem 1 *If a migration matrix $M = (m_{ij})_{i,j=1,...,8}$ is strictly diagonal dominant, i.e., $m_{ii} > \frac{1}{2}$ for every i, then the log-expansion*

$$\tilde{Q}_n \ = \ \sum_{k=1}^{n} (-1)^{k+1} \frac{(M-I)^k}{k} \qquad (n \in \mathbb{N})$$

converges to a matrix $\tilde{Q} = (\tilde{q}_{ij})_{i,j=1,...,8}$ satisfying

1. $\sum_{j=1}^{8} \tilde{q}_{ij} = 0$ *for every* $i = 1, ..., 8$*;*

2. $\exp(\tilde{Q}) = M$*.*

The convergence $\tilde{Q}_n \to \tilde{Q}$ *is geometrically fast.*

Proof. See ISRAEL ET AL. [23]. □

Remark 1 Recall that the generator of a time-continuous Markov chain is given by a so-called Q-matrix $Q = (q_{ij})_{1 \le i,j \le 8}$ satisfying the following properties:

1. $\sum_{j=1}^{8} q_{ij} = 0$ for every $i = 1, ..., 8$;
2. $0 \le -q_{ii} < \infty$ for every $i = 1, ..., 8$;
3. $q_{ij} \ge 0$ for all $i, j = 1, ..., 8$ with $i \ne j$.

For some background on Markov chains we refer to the book by NORIS [31].

The following theorem is a standard result from Markov chain theory:

Theorem 2 *The following two properties are equivalent for a matrix* $Q \in \mathbb{R}^{8\times 8}$*:*

- Q *satisfies Properties 1 to 3 in Remark 1.*
- $\exp(tQ)$ *is a stochastic matrix for every* $t \ge 0$*.*

Proof. See NORIS [31], Theorem 2.1.2. □

Theorem 1, Remark 1 and Theorem 2 open a nice way to construct credit curves which are compatible with Table 1 at the one-year horizon. We start by calculating the log-expansion $\tilde{Q} = (\tilde{q}_{ij})_{i,j=1,...,8}$ of the one-year migration matrix $M = (m_{ij})_{i,j=1,...,8}$ according to Theorem 1. This can be done with a calculation program like Mathematica or Matlab, but can as easily also be implemented in Excel/VBA. Table 3 shows the resulting matrix $\tilde{Q}$.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-7.22%	6.83%	0.20%	0.13%	0.06%	-0.01%	0.00%	0.00%
AA	0.64%	-9.55%	8.32%	0.42%	0.03%	0.11%	0.02%	0.01%
A	0.05%	2.31%	-9.21%	6.23%	0.36%	0.17%	0.03%	0.06%
BBB	0.03%	0.20%	4.91%	-11.99%	5.45%	0.83%	0.31%	0.26%
BB	0.04%	0.09%	0.31%	7.06%	-19.51%	9.47%	1.39%	1.14%
B	-0.01%	0.09%	0.30%	0.22%	6.42%	-20.35%	7.07%	6.25%
CCC	0.14%	-0.02%	0.39%	0.79%	1.81%	14.84%	-55.69%	37.73%
D	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

Table 3: Log-expansion of the modified S&P average one-year migration matrix

Theorem 1 guarantees that $\tilde{Q}$ fulfills Property 1 of generators listed in Remark 1. Condition 2 is also not hurt by $\tilde{Q}$, but Condition 3 is not fulfilled. However, there are only three 'black sheep' in Table 3, namely

- $\tilde{q}_{AAA,B}$ = -1bps,
- $\tilde{q}_{B,AAA}$ = -1bps,
- $\tilde{q}_{CCC,AA}$ = -2bps.

Only these three entries disable $\tilde{Q}$ from being a generator matrix. Because these three values are very small numbers, we feel free to set them equal to zero and decrease the diagonal elements of rows AAA, B and CCC by an amount compensating for the increased row sums, such that at the end the row sums are equal to zero again. In [25] this procedure is called a 'diagonal adjustment'. As a result we obtain a generator matrix $Q = (q_{ij})_{i,j=1,...,8}$ as shown in Table 4.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-7.23%	6.83%	0.20%	0.13%	0.06%	0.00%	0.00%	0.00%
AA	0.64%	-9.55%	8.32%	0.42%	0.03%	0.11%	0.02%	0.01%
A	0.05%	2.31%	-9.21%	6.23%	0.36%	0.17%	0.03%	0.06%
BBB	0.03%	0.20%	4.91%	-11.99%	5.45%	0.83%	0.31%	0.26%
BB	0.04%	0.09%	0.31%	7.06%	-19.51%	9.47%	1.39%	1.14%
B	0.00%	0.09%	0.30%	0.22%	6.42%	-20.35%	7.07%	6.25%
CCC	0.14%	0.00%	0.39%	0.79%	1.81%	14.84%	-55.71%	37.73%
D	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

Table 4: Approximate generator of the modified S&P average one-year migration matrix

From Theorem 1 we know that we get back the original migration matrix M from $\tilde{Q}$ by $\exp(\tilde{Q})$. But what about getting M back from Q? Because we manipulated $\tilde{Q}$ in order to arrive at a generator Q, $\exp(Q)$ will not exactly equal M. What we are interested in now is the error coming from replacing M by $\exp(Q)$. Because the manipulation we did is negligible, we already expect the result of the next proposition.

Proposition 1 *M has an approximate Q-matrix representation by Q. The error is*

$$\|M - \exp(Q)\|_2 \;=\; \sqrt{\sum_{i,j=1}^{8} (m_{ij} - (\exp(Q))_{ij})^2} \;\approx\; 0.000224$$

and therefore negligible. We can safely work with Q instead of $\tilde{Q}$.

Proof. Just calculate the distance. □

In Markov chain theory the achievement of Q is called an *embedding of the time-discrete Markov chain represented by M into a time-continuous Markov chain represented by its generator or Q-matrix Q*. Of course, our embedding only holds in an approximate manner (see Proposition 1), but the error is negligibly small. Probabilists know that the existence of such embeddings is far from being trivial, and to some extent we have been very lucky that it worked so well with the S&P-based migration matrix M. However, there is more than just one technique to find approximate generators; see [24], [23] and [25].

We are basically done now with our credit curves. Our efforts have been rewarded by a generator Q with $\exp(Q) = M$, ignoring the '≈' from now on. The credit curves can be read-off from the collection of matrices $(\exp(tQ))_{t\geq 0}$ by looking at the default columns. More precisely, we obtain

$$p_t^{(R)} \;=\; (e^{tQ})_{i(R),8}$$

where $i(R)$ denotes the transition matrix row corresponding to the given rating R. For the rest of this paper we will always work with the credit curves just constructed. Figure 6 shows a chart of our credit curves from $t = 0$ to $t = 50$ years (quarterly values).
The curves in Figure 6 are typical. Credit curves assigned to subinvestment grade ratings have a tendency to slow-down their growth, because conditional on having survived for some time the chances for further survival improve. For good ratings we see the opposite effect.

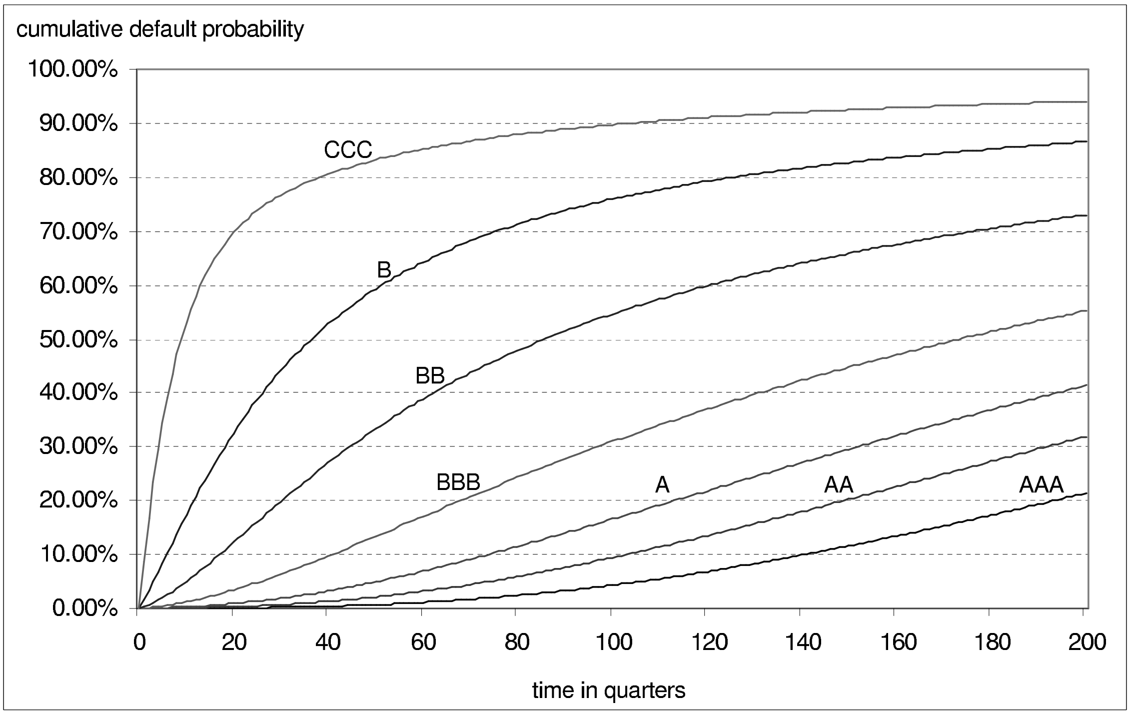

Figure 6: Calibrated curves $(p_t^{(R)})_{t\geq 0}$

4.2 The distribution of single-name default times

If we believe that our credit curves $(p_t^{(R)})_{t\geq 0}$ are correct and really give us the cumulative default probabilities for any given rating R over any time interval $[0, t]$, there is one and only one way to define a default time distribution for an R-rated obligor.

Proposition 2 *Given a credit curve $(p_t^{(R)})_{t\geq 0}$ for a rating R, there exists a unique default times distribution for R-rated obligors.*

Proof. The proof is obvious: Set $F_R(t) = p_t^{(R)}$ for $t \geq 0$. Define a random variable $\tau^{(R)}$ with values in $[0, \infty)$ and distribution function $\mathbb{F}_{\tau^{(R)}}$ by

$$\mathbb{F}_{\tau^{(R)}}(t) \ = \ \mathbb{P}[\tau^{(R)} \leq t] \ = \ F_R(t) \ = \ p_t^{(R)}. \tag{1}$$

For example, $\tau^{(R)} = F_R^{-1}(X)$ with $X \sim U[0, 1]$ will do the job. □

Proposition 2 shows that as soon as our credit curve is established, there is just one way to come up with a default time for an obligor admitting such a credit curve. In the sequel, we continue to use the notation from the proof of Proposition 2. Based on (1), the density of the default time distribution of R-rated obligors can be obtained by calculating the derivative

$$f_{\tau^{(R)}}(t) \ = \ \frac{d}{dt} F_R(t). \tag{2}$$

Figure 7 shows the default time densities w.r.t. our credit curves $(p_t^{(R)})_{t\geq 0}$ for R=AAA, R=AA and R =A. For (single name) default times we expect in general that

- investment grade ratings have a default expectation far in the future, whereas

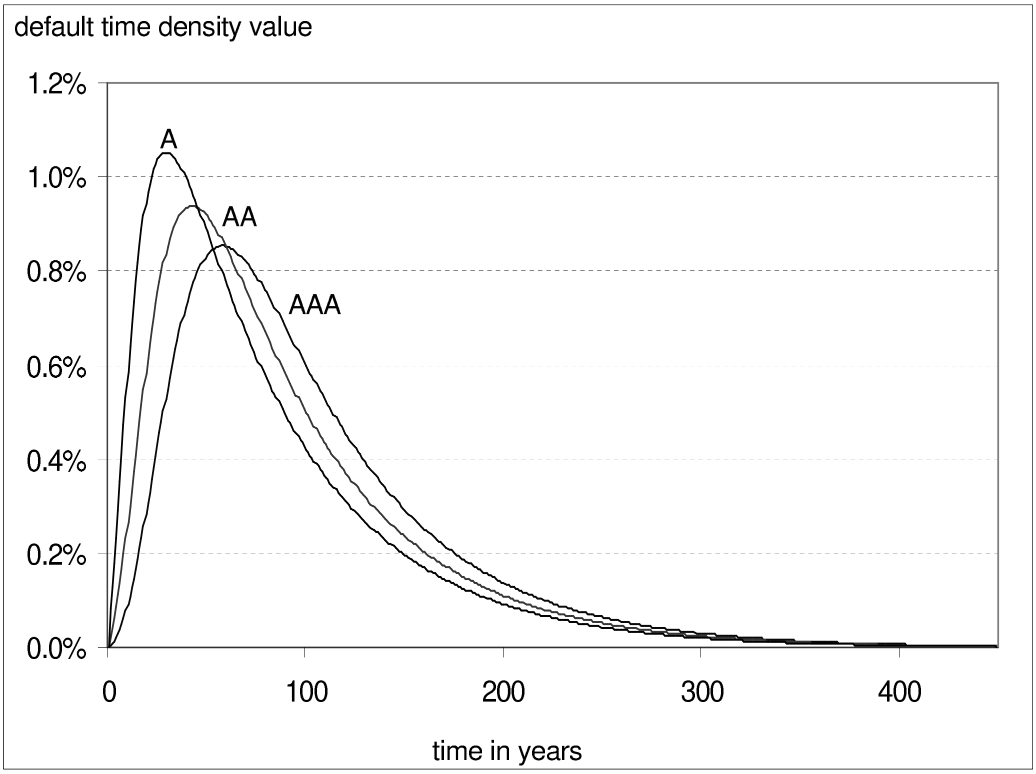

Figure 7: Default time densities w.r.t. $(p_t^{(R)})_{t\geq 0}$ for $R=$AAA,AA,A

- subinvestment grade ratings can be expected to default in the near future.

We will illustrate this by calculating the expectation and standard deviation of the default time distributions calibrated w.r.t. our credit curves. Table 5 reports on our calculation of the mean and standard deviation of $f_\tau^{(R)}$,

$$\mathbb{E}[\tau^{(R)}] \;=\; \int_0^\infty t f_{\tau^{(R)}}(t)dt \qquad \text{and}$$

$$\sigma[\tau^{(R)}] \;=\; \Big(\int_0^\infty (t-\mathbb{E}[\tau^{(R)}])^2 f_{\tau^{(R)}}(t)dt\Big)^{\frac{1}{2}}.$$

in years	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
mean	103	90	80	64	43	25	12
std.dev.	69	68	66	64	56	43	31
std.dev./mean	67%	75%	83%	99%	128%	174%	266%

Table 5: Mean and standard deviation of default times w.r.t. $(p_t^{(R)})_{t\geq 0}$ (time in years)

The numbers show that if one only considers a single credit risky asset, its expected default time typically is far away in the future. Even for the worst credit quality, CCC, the expected time until default is 12 years. However, one should not forget that the chances that a CCC-rated credit defaults within *one year* are almost 30%. This is possible because default time distributions are quite unsymmetric and heavily skewed.

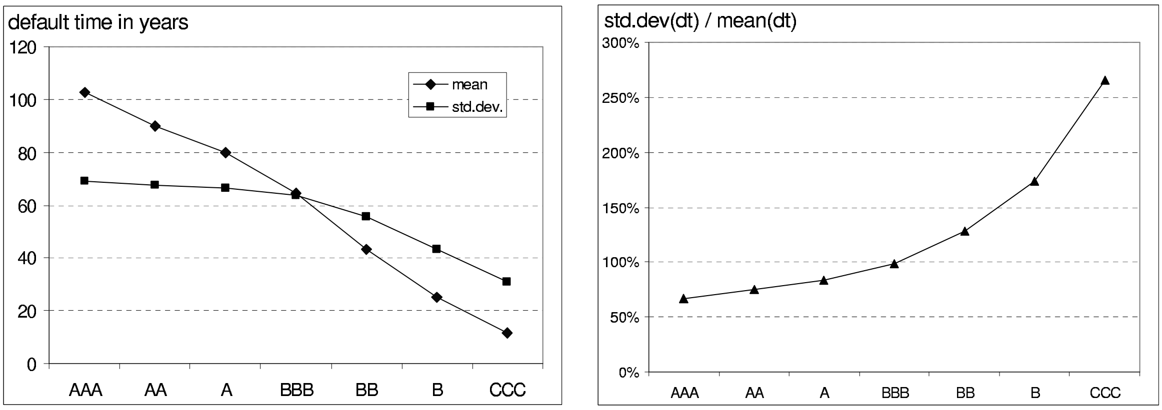

Figure 8: Dependence of default times statistics on ratings

Figure 8 visualizes the dependence of default times means and standard deviations on ratings. It is quite interesting to observe that in the considered case mean and standard deviation coincide exactly at the border between investment and subinvestment grade ratings, namely in BBB. The volatility of default times decreases with decreasing credit quality but increases if considered relative to the corresponding expected default time.

This concludes our discussion of single-name default times. In the next section we are interested in default times for a whole portfolio of credits.

4.3 Multivariate default times distributions

Let us now assume that we look at a portfolio of m obligors. We use the index i to refer to the i-th obligor. The rating assigned to an obligor i will be denoted by $R(i)$. The results from Section 4.2 imply *marginal* default time distributions for our portfolio. We denote the corresponding densities and distribution functions by

$$f_1 = f_{\tau(R(1))},\ \ldots,\ f_m = f_{\tau(R(m))} \qquad \text{and} \qquad \mathbb{F}_1 = \mathbb{F}_{\tau(R(1))},\ \ldots,\ \mathbb{F}_m = \mathbb{F}_{\tau(R(m))}\ .$$

These are the unique (marginal) default time distributions matching our credit curves, and because we decided to believe in our credit curves we will fix these distributions for the rest of this section. What we need now is a method to combine given marginal distributions to a common multivariate distribution reflecting the dependencies between single default times. And we are lucky: there is such a concept known in the statistics literature for many years, namely the copula approach. In fact, the concept of copulas is much older than the history of internal credit risk models. Here in the context of default times one really finds a good example for a meaningful application of copula functions. The problem we started with at the beginning of this section is a classical motivation for the development of copulas. Given m marginal distributions, how can we 'bind' them together to a reasonable multivariate distribution incorporating the dependencies between the m components? Copulas are the answer. Therefore, we now briefly mention some key facts.

Definition 1 *A copula (function) is a multivariate distribution (function) such that its marginal distributions are standard uniform. A common notation for copulas is*

$$C : [0,1]^m \to [0,1], \quad (u_1, \ldots, u_m) \mapsto C(u_1, \ldots, u_m)\ .$$

In our case m always refers to a number of obligors or credit risky instruments.

The following two theorems show that copulas offer a universal tool for constructing and studying multivariate distributions.

Theorem 3 (SKLAR [34], [35]) *For any m-dimensional distribution function $\boldsymbol{F}$ with marginal distributions $F_1, ..., F_m$, there exists a copula function C such that*

$$\boldsymbol{F}(x_1, ..., x_m) \;=\; C(F_1(x_1), ..., F_m(x_m)) \qquad (x_1, ..., x_m \in \mathbb{R}).$$

Moreover, if the marginal distributions $F_1, ..., F_m$ are continuous, then C is unique.

Sketch of Proof. Define the following function on $[0,1]^m$,

$$C(u_1, ..., u_m) \;=\; \boldsymbol{F}(F_1^{-1}(u_1), ..., F_m^{-1}(u_m))\;. \tag{3}$$

Then one only has to verify that C is a copula representing $\boldsymbol{F}$, see also NELSON [29]. □

But more is true. It is not only the case that any multivariate distribution has a copula representation, but copulas can be combined with given marginal distributions.

Proposition 3 *Given a copula C and distribution functions $F_1, ..., F_m$ on $\mathbb{R}$, the function*

$$\boldsymbol{F}(x_1, ..., x_m) \;=\; C(F_1(x_1), ..., F_m(x_m)) \qquad (x_1, ..., x_m \in \mathbb{R})$$

defines a multivariate distribution function with marginal distributions $F_1, ..., F_m$.

Proof. The proof is straightforward. □

It is Proposition 3 we can use to construct multivariate default time distributions. We already found the 'true' marginal distribution functions $\mathbb{F}_1$, ..., $\mathbb{F}_m$ reflecting our credit curves. Choosing a copula function C will yield a multivariate default times distribution

$$\boldsymbol{F}(t_1, ..., t_m) \;=\; C(\mathbb{F}_1(t_1), ..., \mathbb{F}_m(t_m)) \tag{4}$$

for times $t_1, ..., t_m \in [0, \infty)$. So the concept of copulas is a nice tool for constructing multivariate default times distributions. Obviously the challenge now is to find a suitable copula best matching the modeling problem and the underlying data.

Example: Gaussian and Student-t copulas

The most commonly used copula in the context of default times is the Gaussian copula,

$$C[u_1, ..., u_m] \;=\; \boldsymbol{F}_{m,\Gamma}[N^{-1}(u_1), ..., N^{-1}(u_m)] \tag{5}$$

where $N[\cdot]$ denotes the standard normal distribution function, $N^{-1}(\cdot)$ denotes its inverse and $\boldsymbol{F}_{m,\Gamma}$ refers to the multivariate Gaussian distribution function on $\mathbb{R}^m$ with correlation matrix $\Gamma = (\varrho_{ij})_{1\leq i,j\leq m}$ and zero mean. The parameter m refers to the number of obligors in the considered portfolio. Gaussian copulas and their application to CDOs have been studied in various papers, see [20], [26], [33], just to mention a few already quoted research papers. One safely can say that whenever people are not explicitly addressing the problem of copula selection, they most often rely on a Gaussian copula.

However, there are many alternatives. The non-Gaussian copula applied most often definitely is the Student-t copula, defined by

$$C[u_1, ..., u_m] \;=\; \boldsymbol{F}_{t(n,\Gamma)}[F_{t(n)}^{-1}(u_1), ..., F_{t(n)}^{-1}(u_m)] \tag{6}$$

where $\boldsymbol{F}_{t(n,\Gamma)}$ denotes the multivariate t-distribution function with n degrees of freedom and (linear) correlation matrix $\Gamma \in \mathbb{R}^{m\times m}$, and $F_{t(n)}$ denotes the t-distribution function with n degrees of freedom. Note that the $t(n,\Gamma)$-distribution can be derived by choosing a Gaussian vector $\boldsymbol{Y} = (Y_1, ..., Y_m) \sim N(0,\Gamma)$, a $\chi^2(n)$-distributed random variable X, independent of $\boldsymbol{Y}$, and considering the product $\sqrt{n/X}\,\boldsymbol{Y}$, which will be $t(n,\Gamma)$-distributed.

There is a whole universe of copulas available in the stochastics literature; see for example EMBRECHTS ET AL. [17] and BOUYÉ ET AL. [15]. Making the 'right' choice of copula is far from being trivial. In the next section, we briefly indicate what difference the choice of copulas makes to a CDO model; see also Table 7 and Figure 12.

Copula impact on CDO performance: Gaussian versus Student-t

If the degrees of freedom are large enough, the difference between Gaussian und t-copulas will vanish if they rely on the same linear correlation matrix Γ. This is due to the fact that for large n $t(n)$ is approximately normal. Moreover, the multivariate t-distribution inherits its linear correlation from the correlation matrix Γ of the involved Gaussian vector,

$$\mathrm{Corr}[\sqrt{n/X}\,Y_i, \sqrt{n/X}\,Y_j] \;=\; \mathrm{Corr}[Y_i, Y_j]$$

using the same notation as in the previous section. But if one decreases the degrees of freedom, the t-copula will show more and more tail dependency; see EMBRECHTS ET AL. [17], FREY and MCNEIL [18], and [13], pages 106-112. This is graphically illustrated in Figure 9. In this example, the chosen linear correlation equals $\varrho = 0.4$.

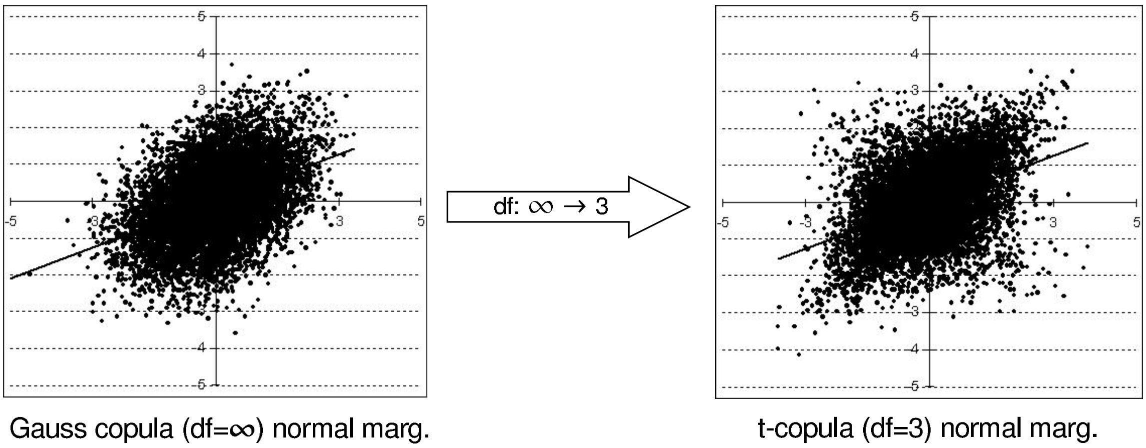

Figure 9: Increasing 'tail dependency' by decreasing the degrees of freedom

For CDO modeling, this insight has important consequences. *More tail dependency typically means a higher potential for joint defaults which in turn implies higher stress for senior tranches in CDO transactions.* In other words, changing the copula used for our default times model means changing the economics of different tranches in different ways; see Section 5. On one side this observation shows that the choice of a particular copula function induces a high model risk, but on the other side it provides a very useful tool for stress testing different CDO tranches. Stress testing is a must in CDO modeling anyway, because most often there is a significant amount of uncertainty on the parameter side involved. *It is very natural to make copulas a part of the stress testing program for CDO tranche investments.*

A remark on asset value models and default times

It is possible to calibrate a default times model *compatible in distribution with a one-period asset value model* (e.g., period = one year) according to BLACK and SCHOLES [12] and MERTON [28]; see [20], [26], [33]. The basic idea is as follows. We start with geometric Brownian motions

$$A_t^{(i)} \;=\; A_0^{(i)} \exp[(\mu_i - \tfrac{1}{2}\sigma_i^2)t + \sigma_i B_t^{(i)}] \qquad (i = 1, ..., m;\; t \in [0, T]) \tag{7}$$

where the Brownian motions are correlated and admit a correlation matrix $\Gamma = (\varrho_{ij})_{1\le i,j\le m}$ constant over time. The processes $(A_t)_{t\ge 0}$ are interpreted to give the asset value of the corresponding obligors at any time. Asset value processes can not be observed in the market, but can be inferred from equity processes, see CROSBIE [16], NICKEL ET AL. [30], and [13], 3.4. The link to default risk is given by a latent variables approach by means of a Bernoulli mixture model, where joint default probabilities can be written in the following general form; see JOE [21], FREY and MC NEIL [18], and [13], 2.1. For $\delta_1, ..., \delta_m \in \{0, 1\}$,

$$\mathbb{P}[L_1 = \delta_1, ..., L_m = \delta_m] \;=\; \int\limits_{[0,1]^m} \prod_{i=1}^{m} p_i^{\delta_i}(1 - p_i)^{1-\delta_i} d\mathbb{F}(p_1, ..., p_m).$$

Here, the complexity of the model is completely hidden in the distribution function $\mathbb{F}$. For example, $\mathbb{F}$ in its most simple form is applied in so-called uniform portfolio models, where all obligors are correlated the same way, admitting a uniform default probability. In this case (see FINGER [19], VASICEK [37], and [13], 2.1.2), the distribution function $\mathbb{F}$ is given by

$$\mathbb{F} \;=\; N \circ g^{-1}, \qquad g(y) \;=\; N\Big[\frac{N^{-1}(\text{PD}) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\Big] \quad (y \in \mathbb{R}), \tag{8}$$

where $N[\cdot]$ denotes the distribution function of the standard normal distribution, ϱ refers to the uniform asset correlation between obligors, and PD denotes the (one-period, e.g., one year) uniform default probability of the portfolio. The probability for k out of m defaults can then be written as

$$\mathbb{P}[L_1 + \cdots + L_m = k] \;=\; \binom{m}{k} \int\limits_{-\infty}^{\infty} g(y)^k (1 - g(y))^{m-k} dN(y).$$

Note that the function $g(y)$ from Formula (8) represents the 'heart' of the *Basel II benchmark risk weights* in the new capital accord.

Now fix some horizon $T > 0$. Coming back to the general case of geometric Brownian asset value processes, the Bernoulli variable indicating default of obligor i over time T typically is defined by

$$L_i^{(T)} \;=\; \mathbf{1}_{\{A_T^{(i)} \le \tilde{c}_T^{(i)}\}} \tag{9}$$

where $\tilde{c}_T^{(i)}$ denotes the default-critical threshold for obligor i, the so-called default point; see [16]. The link to our credit curves $(p_t^{(R)})_{t\ge 0}$ from Section 4.1 is given by a calibration of the default point $\tilde{c}_T^{(i)}$ in a way reflecting the T-year cumulative default probability,

$$\mathbb{P}[L_i^{(T)} = 1] \;=\; \mathbb{P}[A_T^{(i)} \le \tilde{c}_T^{(i)}] \;\stackrel{!}{=}\; p_T^{(i)} \tag{10}$$

where $p_T^{(i)} = p_T^{(R(i))}$. Based on a simple reformulation of (7), we obtain the *distributional* equation

$$\mathbb{P}[L_i^{(T)} = 1] \;=\; \mathbb{P}[B^{(i)} \le c_T^{(i)}] \qquad \text{with} \qquad c_T^{(i)} = \frac{\ln(\tilde{c}_T^{(i)}/A_0^{(i)}) - (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}\;. \tag{11}$$

where $B^{(i)} \sim B_1^{(i)} \sim N(0,1)$. Note that conceptually $B^{(i)}$ has nothing but the standard normal distribution in common with the value $B_1^{(i)}$ of the driving Brownian motion. The *time-dynamics* of the underlying process is not taken into account in this 'fixed-horizon' approach. However, due to $B^{(i)} \sim N(0,1)$ and Equations (10) and (11) we can conclude that

$$c_T^{(i)} \;=\; N^{-1}(p_T^{(i)}) \;=\; N^{-1}(F_i(T)) \tag{12}$$

for the default point of obligor i, where $(F_i(t))_{t\geq 0}$ denotes the credit curve of that obligor. The second '=' in (10) additionally 'forces' that at the horizon T

$$\mathbb{P}[L_i^{(T)} = 1] \;=\; p_T^{(i)} \;=\; \mathbb{P}[\tau_i \leq T] \qquad \text{with} \qquad \tau_i \sim \mathbb{F}_i \tag{13}$$

where $\mathbb{F}_i$ denotes the default time distribution function of obligor i; see Section 4.2. We have

$$\mathbb{P}[\tau_i \leq T] \;=\; \mathbb{P}[L_i^{(T)} = 1] \;=\; \mathbb{P}[B^{(i)} \leq N^{-1}(F_i(T))] \;=\; \mathbb{P}[F_i^{-1}(N[B^{(i)}]) \leq T].$$

This motivates the current 'standard approach' to correlated default times, defining

$$\tilde{\tau}_i \;=\; F_i^{-1}(N[B^{(i)}]) \quad (i = 1, ..., m) \qquad \text{where} \quad (B^{(1)}, ..., B^{(m)}) \sim N(0, \Gamma) \tag{14}$$

is multivariate Gaussian with correlation matrix Γ; see [20], [26], [13], 7.3. By construction we have $\tilde{\tau}_i \sim \tau_i$, so (14) shows a way to define the default time of obligor i as a function of a variable in distribution equal to the (standardized) one-year asset value log-return $B_1^{(i)}$ of obligor i. Naturally, default times defined according to Equation (14) will inherit the dependence structure of the involved one-period asset value model.

Summarizing, (14) yields *correlated default times marginally matching given credit curves and inheriting the dependence structure of a given one-period asset value model.*

Moreover, the cumulative default distribution up to time T arising from the default times approach and the distribution of the portfolio defaults arising from a one-period (more precisely, the period is the time interval $[0, T]$) asset value model coincide,

$$\sum_{i=1}^{m} \mathbf{1}_{\{\tilde{\tau}_i \leq T\}} \;\sim\; \sum_{i=1}^{m} L_i^{(T)}$$

for any fixed horizon T. Note that this relation refers to equality in distribution only.

However, in order to define correlated default times in line with (14), it is not really necessary to think in terms of an asset value model, because they simply can be derived by *combining our single default times distributions from Section 4.2 by means of a Gaussian copula.*

More precisely, combining (marginal) default time distribution functions $\mathbb{F}_1, ..., \mathbb{F}_m$ by means of a Gaussian copula according to (5), we get a multivariate default times distribution function

$$\boldsymbol{F}(t_1, ..., t_m) \;=\; \boldsymbol{F}_{m,\Gamma}[N^{-1}(\mathbb{F}_1(t_1)), ..., N^{-1}(\mathbb{F}_m(t_m))].$$

For the corresponding default times $\tau_1, ..., \tau_m$ we derive at the following equation,

$$\begin{aligned}
&\mathbb{P}[\tau_1 \leq t_1, ..., \tau_m \leq t_m] \;=\; \boldsymbol{F}_{m,\Gamma}[N^{-1}(\mathbb{F}_1(t_1)), ..., N^{-1}(\mathbb{F}_m(t_m))] \\
&=\; \mathbb{P}[X_1 \leq N^{-1}(\mathbb{F}_1(t_1)), ..., X_m \leq N^{-1}(\mathbb{F}_m(t_m))] \\
&=\; \mathbb{P}[\mathbb{F}_1^{-1}(N[X_1]) \leq t_1, ..., \; F_m^{-1}(N[X_m]) \leq t_m] \qquad \text{where} \quad (X_1, ..., X_m) \sim N(0, \Gamma).
\end{aligned}$$

Going back to (14) we find that $(\tau_1, ..., \tau_m) \sim (\tilde{\tau}_1, ..., \tilde{\tau}_m)$.

Altogether the derivation of default times according to (14) explains why marginal default time distributions are often combined to a multivariate default times distribution by use of a Gaussian copula. The reason is that a one-period asset value model in the classical setting as introduced by MERTON matches a default times model relying on a Gaussian copula. However, *in the same way as other processes than Brownian motions are used in option pricing theory today, other than Gaussian copulas become more and more popular for default times distributions. The dependence structure of a non-normal asset value model then can be carried over to the default times copula in an analogous way as exercised for the Gaussian copula case in* (14).

Barrier diffusion models and first passage times

In this section we briefly want to mention other approaches to default times by means of barrier models in order to give the reader some motivation to investigate other promising ideas applicable to CDO analysis. A more detailled exposition and discussion of the results indicated in this section can be found in [14].

For example, OVERBECK and SCHMIDT [32] start with (marginal) default time distributions in a similar way as we did in 4.2 by writing

$$\mathbb{P}[\tau_i < t] \;=\; \mathbb{F}_i(t) \qquad (t \geq 0;\; i = 1, ..., m). \tag{15}$$

Hereby, the distribution functions $\mathbb{F}_i(t)$ are given from external sources, e.g., credit curves as in Section 4.1. Additionally they define the pairwise joint default probabilities (JDP) w.r.t. a fixed horizon T by

$$\text{JDP}_{ij} \;=\; \mathbb{P}[\tau_i < T, \tau_j < T]. \tag{16}$$

The problem they study is the following:

Problem 1 *Given* $\mathbb{F}_1, ..., \mathbb{F}_m$ *and* $(JDP_{ij})_{1 \leq i,j \leq m}$*, find stochastic processes* $(X_t^{(i)})_{t \geq 0}$ *and barriers* $(c_t^{(i)})_{t \geq 0}$*,* $i = 1, ..., m$*, such that*

$$\tau_i \;=\; \inf\{t \geq 0 \;:\; X_t^{(i)} \leq c_t^{(i)}\},$$

where τ_i *means the default time of obligor* i *satisfying (15) and (16).*

Problem 1 searches for a *first passage* or *hitting time* matching the default time of obligors admitting a prescribed credit curve and prescribed default event correlations, because the JDPs are directly related to default correlations and vice versa; see [13], pages 58-61.

The key idea elaborated in [32] for tackling Problem 1 is to define a time scale transformation T_i for every asset and to transform correlated Brownian motions $(B_t^{(i)})_{t \geq 0}$,

$$X_t^{(i)} \;=\; B_{T_i(t)}^{(i)} \qquad (t \geq 0;\; i = 1, ..., m),$$

then applied for the definition of first passage times w.r.t. barriers $(c_t^{(i)})_{t \geq 0}$,

$$\tilde{\tau}_i \;=\; \inf\{t \geq 0 \;:\; X_t^{(i)} \leq c_t^{(i)}\}. \tag{17}$$

The processes $(X_t^{(i)})_{t \geq 0}$ are called *ability-to-pay* processes. In [32], Proposition 1, they define a suitable time scale transformation T_i such that $\tilde{\tau}_i$ coincides with the default time τ_i defined by means of the prescribed credit curves (15). The proof is based on the well-known first passage time distribution for Brownian motion. In a second part of the paper ([32], 4.2) it is shown that the time changed Brownian motions can be calibrated w.r.t. given JDPs. Based on a paper by ZHOU [38], the JDPs can be described in closed analytic form as a function of the correlation of the underlying Brownian motions.

The results in [32] provide an intuitive way (time-transforming Wiener processes) to derive correlated default times by means of a barrier diffusion model. In HULL and WHITE [22], a comparable analysis can be found. Regarding default barrier models we refer to ALBANESE ET AL. [6] and AVELLANEDA and ZHU [8].

4.4 Analytic approximations

For some transactions, analytic or semi-analytic approximations can be applied in order to speed-up the evaluation of CDOs. Whether analytic shortcuts can be applied or not strongly depends on the structure of the transaction and the underlying asset pool.

Analytic derivation of expected loss and PD of a tranche

The most typical example where an analytic approach is as good as any Monte-Carlo simulation approach is the case of a synthetic (balance sheet motivated) transaction for the purpose of regulatory capital relief and risk transfer, referenced to a large homogeneous pool of reference assets, e.g., a large portfolio of retail loans or a highly diversified portfolio of SMEs. More precisely, for the sequel we consider (as an example) a structure satisfying the following conditions:

1. The underlying reference pool is highly diversified and can be (approximately) represented by a uniform reference portfolio with infinite granularity.

2. Amortization of notes on the liability side follows sequentially top-down in decreasing order of seniority (highest seniority tranche first, second highest seniority tranche next, and so on).

3. Losses are written-off sequentially bottom-up in increasing order of seniority (equity tranche bears the first loss, first junior piece bears the second loss, and so on).

4. CDO notes are referenced to the underlying pool of assets (e.g., by credit linked notes). Interest payments on notes will be paid by the originator in an amount of

$$\text{Interest(Tranche)} \;=\; \text{Volume(Tranche)} \times [\text{LIBOR} + \text{Spread(Tranche)}]. \tag{18}$$

   Besides loss write-offs (bottom-up; see Condition 3), no other random events trigger repayments or interest payments.

Condition 4 implies that a default on interest payment obligations can only happen if the *originator* defaults on its financial/contractual obligation, because the promise made to notes investors is to pay interest according to (18) unconditionally. However, due to loss write-offs, the *nominal* amount of interest payments can decrease if a tranche's volume decreases as a consequence of losses.

For the sequel we assume without loss of generality a *fixed LGD of 100%.* By scaling and a substitution in the respective formulas, any fixed LGD can be implemented, replacing any gross loss L by a realized net loss $L \times \text{LGD}$. As another simplifying condition, we assume that the asset pool is static (non-managed) and has a bullet exposure profile until maturity. However, by

'WAL-adjustments' (WAL stands for weighted average life) the results of this section also apply to amortizing asset pools.

For transactions as the one described above, the risk of tranches can be quantified by a closed-form analytic approach. Condition 1 allows to replace the original reference pool by a uniform or homogeneous portfolio with infinite granularity admitting a uniform credit curve $(p_t)_{t\geq 0}$ as calibrated in Section 4.1 and a uniform asset correlation ϱ. Let us assume that the considered CDO matures at time T. Then, the cumulative loss L at the horizon T is given by (cp. (8))

$$L \;=\; p(T,Y) \;=\; N\Big[\frac{N^{-1}(p_T) - \sqrt{\varrho}\,Y}{\sqrt{1-\varrho}}\Big] \qquad \text{where} \quad Y \sim N(0,1). \tag{19}$$

The derivation of this representation is well-known, due to VASICEK [37], and can be found in a more general setting, e.g., in [13], pages 87-94. The expression on the right side in equation (19) is the loss variable of a portfolio with infinitely many obligors (limit case) where all obligors have a PD of p_T and are pairwise correlated with an asset correlation ϱ. The variable Y has an interpretation as a macro-economic factor driving the loss of the portfolio. Because $p(T,Y)$ corresponds to a portfolio of infinitely many assets, idiosyncratic risk has been completely removed by diversification, so that the randomness of Y is the sole source of the riskyness of the portfolio loss $p(T,Y)$.

In the sequel we will exploit the absolute continuity of $p(T,Y)$ by relying on its density

$$f_{p_T,\varrho}(x) \;=\; \sqrt{\frac{1-\varrho}{\varrho}}\,\exp\Big(\frac{1}{2}\left(N^{-1}(x)\right)^2 - \frac{1}{2\varrho}\left(N^{-1}(p_T) - \sqrt{1-\varrho}\,N^{-1}(x)\right)^2\Big) \tag{20}$$

see [13], page 91. By construction, we get back the credit curve $(p_t)_{t\geq 0}$ by calculating expectations,

$$p_t \;=\; \mathbb{E}[p(t,Y)] \;=\; \int_{-\infty}^{\infty} N\Big[\frac{N^{-1}(p_t) - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}\Big]dN(y) \;=\; \int_0^1 x f_{p_t,\varrho}(x)dx,$$

assuming that the severity of loss equals 100% in case of defaults.

Now, any tranching consisting of q tranches on the liability side of the CDO can be written as a partition of $[0,1)$ in the following way,

$$\Theta_i \;=\; [\alpha_i, \alpha_{i+1}) \qquad (i = 1,...,q;\; 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{q+1} = 1).$$

Hereby we assume losses to be normalized (in percentage) to the unit interval (0 = no loss, 1 = full loss of the portfolio's total notional amount). Given the simple cash flow structure assumed at the beginning of this section, the loss L_i of tranche Θ_i is given by

$$L_i \;=\; \Lambda_i(L) \;=\; \min[\max(0, L-\alpha_i), \alpha_{i+1}-\alpha_i] \qquad (i = 1,...,q) \tag{21}$$

where L represents the portfolio loss at time T, $L = p(T,Y)$.

Proposition 4 *For a CDO with maturity T satisfying the conditions listed at the beginning of this section, the expected loss of tranche Θ_i, normalized to the tranche size, can be calculated by*

$$\mathbb{E}[L_i] \;=\; \frac{1}{|\Theta_i|}\int_0^1 \Lambda_i(x) f_{p_T,\varrho}(x)dx$$

where Λ_i is the function defined in (21) and $|\Theta_i| = \alpha_{i+1} - \alpha_i$.

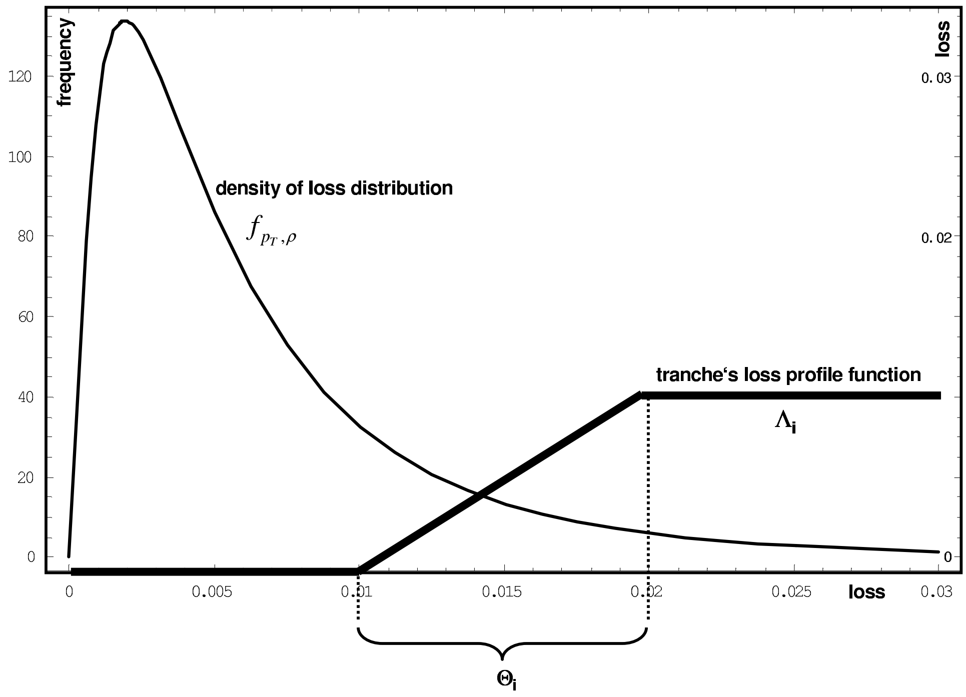

Figure 10: Loss profile function of a CDO tranche

Proof. The assertion of the proposition is obvious. □

Proposition 4 offers a closed-form expression for the expected loss of a tranche, illustrated by Figure 10. The hitting probability of tranches also can be expressed in closed form.

Proposition 5 *Under the stated conditions, the probability π_i that tranche Θ_i is hit by a loss equals*

$$\pi_i \;=\; 1 - N\Big[\frac{1}{\sqrt{\varrho}}\Big(N^{-1}(\alpha_i)\sqrt{1-\varrho} - N^{-1}(p_T)\Big)\Big]\;.$$

Remember that α_i denotes the lower boundary of tranche Θ_i.

Proof. First of all note that we have

$$\mathbb{P}[p(T,Y) \le x] \;=\; \mathbb{P}\Big[-Y \le \frac{N^{-1}(x)\sqrt{1-\varrho} - N^{-1}(p_T)}{\sqrt{\varrho}}\Big]$$

for all $x \in [0,1]$ according to (19). Taking $Y \sim N(0,1)$ into account and considering

$$\pi_i \;=\; \mathbb{P}[p(T,Y) > \alpha_i] \;=\; 1 - \mathbb{P}[p(T,Y) \le \alpha_i],$$

the proof of the proposition follows. □.

Proposition 5 offers a way to calculate the PD of a CDO tranche and Proposition 4 helps calculating its expected loss (EL). Then, the loss given default (LGD) can be defined and calculated by

$$\mathrm{LGD}_i \;=\; \frac{\mathbb{E}[L_i]}{\pi_i} \tag{22}$$

for any tranche Θ_i, $i = 1, ..., q$. In this way, the three main components of basic risk analysis (PD, EL and LGD) are fully specified. Another interesting application of Proposition 5 is the derivation of a model-based *implied rating* of a CDO tranche.

An illustrative example

We conclude this section by an example applying the results just derived. Let us consider a CLO satisfying the assumptions stated at the beginning of Section 4.4. The assumed tranching of the CLO is reported in Figure 11. The maturity of the CLO is in 10 years, counted from today on.

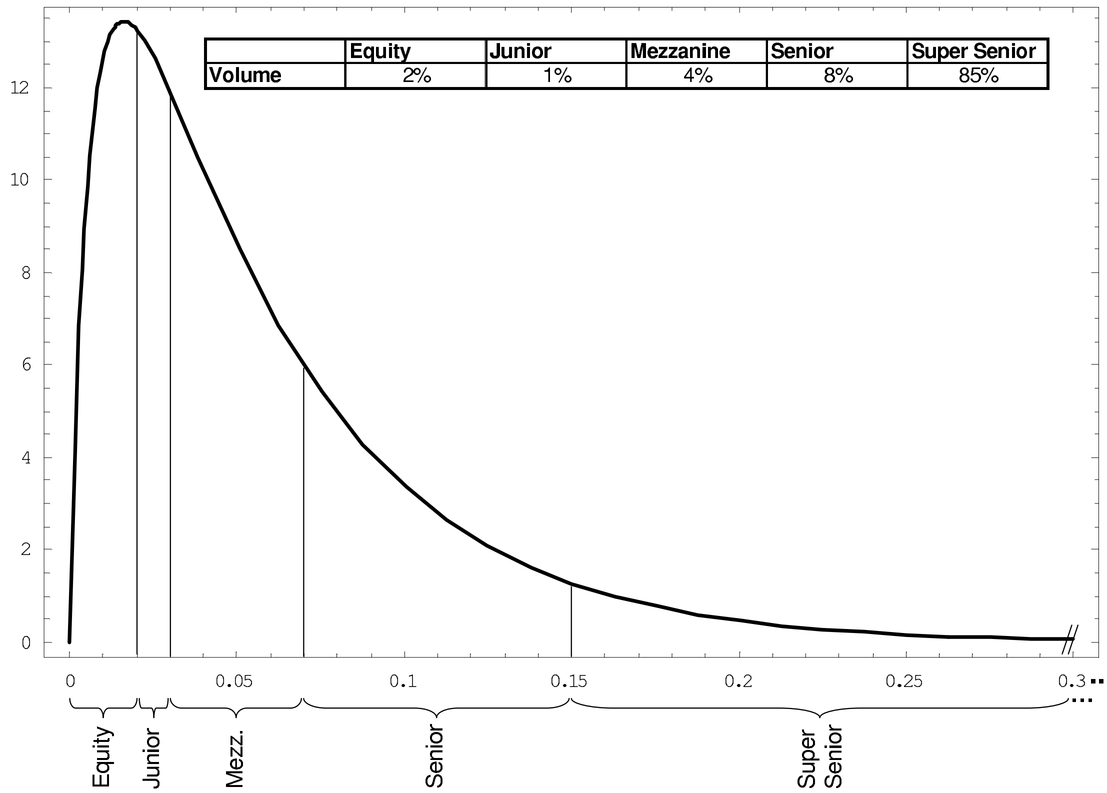

Figure 11: Tranching of a portfolio's loss distribution

Let us assume that the underlying reference pool (static, bullet profile) can be (approximately) replaced by a uniform portfolio with a BBB-credit curve $(p_t)_{t \geq 0}$, see Figure 6, and a uniform asset correlation $\varrho = 20\%$. For $T = 10$ we then obtain $p_1 = 36$bps and $p_T = 9.8\%$. The cumulative loss distribution for p_T, ϱ and an assumed fixed LGD of 60% (i.e., an overall recovery of 40%) is plotted in Figure 11. The cumulative EL of the portfolio equals 5.88%.

We now apply Propositions 4 and 5 to our sample transaction and obtain Table 6 as a result. Here are some comments:

1. **Equity tranche**: In the analytic approach, the PD of the equity tranche typically is 100% because the loss distribution is absolutely continuous, implying that $\mathbb{P}[L = 0] = 0$.

2. **Rule-of-thumb for LGDs**: In general, the LGD of a tranche is lower for thick tranches and higher for thin tranches. To illustrate this, assume that a tranche consists of one point only, say, lower and upper boundaries equal some $\alpha \in (0, 1)$. Then, as soon as a loss hits the tranche, it will completely be wiped out the same moment. Therefore, its LGD is 100%. Now set $\beta = \alpha$, keep α as the lower boundary of the tranche fixed, but increase β as the upper boundary of the tranche. The higher β, the longer it takes for losses to eat into the tranche before they have consumed all of the tranche's capital. The LGD's rule-of-thumb is reflected by the results in Table 6.

3. **Super senior tranche**: The cumulative PD of the super senior tranche equals 6.15%. This is very high for a super senior swap and due to the illustrative character of the example. In typical 'real life' transactions we considered, the super senior swap's PD never exceeded a few basispoints. However, even in our illustrative sample tranching one observes that the cumulative 10-year EL is quite low, also reflected by the small LGD of the tranche. A linear annualization of the EL would yield an annual EL of about 3bps.

	Volume	cumul. PD	cumul. EL	LGD
Equity	2.00%	100.00%	90.73%	90.73%
Junior	1.00%	78.13%	71.69%	91.76%
Mezzanine	4.00%	65.48%	46.03%	70.30%
Senior	8.00%	30.58%	15.35%	50.20%
Super Senior	85.00%	6.15%	0.33%	5.37%

Table 6: Result of applying Propositions 4 and 5 to our example

Obviously, the overall expected loss of the portfolio can be obtained by calculating

$$\sum_{i=1}^{5} \text{Volume}(\text{Tranche}_i) \times \text{EL}(\text{Tranche}_i),$$

Indeed, doing the calculation yields the portfolio's cumulative EL of 5.88%. In other words, the portfolio's EL has been allocated to CDO tranches in an *additive* way as expected.

4.5 Semi-analytic techniques

We now come to a semi-analytic approach applicable to a much broader class of transactions than the purely analytic approach explained in the previous section. Here, we

- only make the assumption that the underlying reference pool is highly diversified and can be approximately represented by a synthetic homogeneous reference pool,
- but allow for all kinds of cash flow elements in the structural definition of the CDO.

In such cases, the semi-analytic technique is a powerful tool to quickly evaluate a CDO.

The approach works as follows. Instead of considering the default times τ_i of single obligors, we consider the *fraction of obligors with a default time within the considered payment period*. To make this precise, denote by τ_i the default time of obligor i and by $L^{(m)}$ the cumulative loss for a portfolio of m obligors over quarterly payment periods $1, ..., T$, where T refers to the maturity of the CDO. The exposure outstanding on loan i in period/quarter j will be denoted by $E_{i,j}$. We assume the following natural conditions, considering an increasing number of obligors in the portfolio:

1. The exposures in the portfolio do not increase over time, i.e., $E_{i,j-1} \geq E_{i,j}$ for all $i = 1, ..., m$, $j = 2, ..., T$, and $m \in \mathbb{N}$, $m \uparrow \infty$.

2. The total exposure at time j,

$$E_j^{(m)} = \sum_{i=1}^{m} E_{i,j},$$

converges to a limit relative to the portfolio's start exposure,

$$\lim_{m\to\infty} \frac{E_j^{(m)}}{E_1^{(m)}} \;=\; w_j$$

for every fixed payment period $j = 1, ..., T$. Due to Condition 1, $w_j \in [0, 1]$.

3. With increasing number of obligors, the total exposure of the portfolio strictly increases to infinity for all fixed payment periods j, i.e., $E_j^{(m)} \uparrow \infty$ for $m \uparrow \infty$ for every $j = 1, ..., T$.

4. With increasing number of obligors, exposure weights shrink very rapidly,

$$\sum_{m=1}^{\infty} \Big(\frac{E_{m,j}}{E_j^{(m)}}\Big)^2 \;<\; \infty$$

for every payment period $j = 1, ..., T$.

These conditions are sufficient but obviously not necessary for establishing the following results. More relaxed conditions are easy to formulate. Here we only illustrate the basic principle.

An example for exposures satisfying Condition 2 is the case of uniform amortization profiles,

$$\exists \quad 1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_T \geq 0 \quad \forall\, i, j : \; \frac{E_{i,j}}{E_{i,1}} \;=\; w_j \;. \tag{23}$$

To give an example, this condition is fulfilled in collateralized swap obligations where on the asset side protection is sold for m single reference names, typically in form of a 5-year bullet profile of m (equal amount) CDSs, and on the liability side protection is bought on the (diversified) pool of CDS in form of tranched securities with a suitable leverage regarding spreads on volumes. In this particular case, $w_j = 1$ for all j.

An example for exposures fulfilling Conditions 3 and 4 is the case where the exposures are captured in a band, $0 < a \leq E_{i,j} \leq b < \infty$ for all i, j. Then, we get for Condition 3,

$$E_j^{(m)} \;=\; \sum_{i=1}^{m} E_{i,j} \;\geq\; m \times a \;\uparrow\; \infty,$$

and for Condition 4,

$$\sum_{m=1}^{\infty} \Big(\frac{E_{m,j}}{E_j^{(m)}}\Big)^2 \;\leq\; \sum_{m=1}^{\infty} \frac{b^2}{m^2 a^2} \;=\; \frac{b^2}{a^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \;<\; \infty.$$

Conditions 1-4 are not really restrictive and will be satisfied in most cases.

We fix Conditions 1-4 for the rest of this section. Assuming for a moment an LGD of 100% and zero collateral, the percentage cumulative loss for an m-obligor portfolio is

$$L^{(m)} \;=\; \sum_{j=1}^{T} \frac{E_j^{(m)}}{E_1^{(m)}} X_j^{(m)} \;, \quad \text{with} \quad X_j^{(m)} = \sum_{i=1}^{m} w_{i,j}^{(m)} \mathbf{1}_{\{j-1 \leq 4\tau_i < j\}} \tag{24}$$

where $w_{i,j}^{(m)}$ denotes the exposure weight for obligor i in payment period j,

$$w_{i,j}^{(m)} \;=\; \frac{E_{i,j}}{E_j^{(m)}} \qquad (i = 1, ..., m;\; j = 1, ..., T).$$

Note that here we write '4τ' instead of 'τ' because we consider quarterly payment periods w.r.t. a time variable t counting in years. A generalization to other payment frequencies is straightforward. Conditions 3 and 4 in the above list are essentially 'cloned' from [13], Assumption 2.5.2, in order to establish Proposition 7. But before, we need to state

Proposition 6 *For a homogeneous portfolio with a credit curve $(p_t)_{t\geq 0}$ and a uniform asset correlation ϱ, the probability that obligor i defaults in the time period $[s, t)$ with $s < t$ conditional on $Y = y$ is given by*

$$\mathbb{P}[s \leq \tau_i < t \mid Y = y] \;=\; N\Big[\frac{N^{-1}(p_t) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\Big] - N\Big[\frac{N^{-1}(p_s) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\Big].$$

Proof. From Equation (19) we conclude

$$\mathbb{P}[\tau_i < t \mid Y = y] \;=\; N\Big[\frac{N^{-1}(p_t) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\Big]\;.$$

This immediately implies the assertion of the proposition. □

Proposition 7 *Under the conditions of this section, for a homogeneous portfolio with a credit curve $(p_t)_{t\geq 0}$ and uniform asset correlation ϱ we obtain*

$$\mathbb{P}\Big[\lim_{m\to\infty}\Big[X_j^{(m)} - \Big(N\Big[\frac{N^{-1}(p_{\frac{j}{4}}) - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}}\Big] - N\Big[\frac{N^{-1}(p_{\frac{j-1}{4}}) - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}}\Big]\Big)\Big] = 0\Big] = 1,$$

where $Y \sim N(0,1)$. Recall that in this section we consider quarterly payment periods.

Proof. The proof is a straightforward modification of the argument provided in [13], pages 88-89, but for the convenience of the reader we provide the argument. The usual 'trick' to prove such results is to condition on the factor Y. We write $\mathbb{P}_y = \mathbb{P}[\,\cdot\,|Y{=}y]$ for the conditional probability measures. Fix $y \in \mathbb{R}$. Then, the random variables

$$Z_{i,j} \;=\; E_{i,j}\mathbf{1}_{\{j-1\leq 4\tau_i<j\}} - \mathbb{E}[E_{i,j}\mathbf{1}_{\{j-1\leq 4\tau_i<j\}} \mid Y] \qquad (i = 1, ..., m;\; j = 1, ..., T)$$

are i.i.d. w.r.t. $\mathbb{P}_y$ and centered. The sequence $(E_j^{(k)})_{k=1,2,...}$ is strictly increasing to infinity due to Condition 3 for any $j = 1, ..., T$. Moreover,

$$\sum_{k=1}^{\infty}\frac{1}{(E_j^{(k)})^2}\,\mathbb{E}_y[Z_{k,j}^2] \;\leq\; \sum_{k=1}^{\infty}\frac{1}{(E_j^{(k)})^2}\,4\,E_{k,j}^2 \;=\; 4\sum_{k=1}^{\infty}\Big(\frac{E_{k,j}}{E_j^{(k)}}\Big)^2 \;<\; \infty\;.$$

due to Condition 4. Then a version of the strong law of large numbers based on Kronecker's Lemma (see, e.g., [9]) implies that

$$\lim_{m\to\infty}\frac{1}{E_j^{(m)}}\sum_{i=1}^{m} Z_{i,j} \;=\; 0 \qquad \mathbb{P}_y\text{-almost surely.}$$

From this we conlude for every $y \in \mathbb{R}$

$$\mathbb{P}[\lim_{m\to\infty}(X_j^{(m)} - \mathbb{E}[X_j^{(m)} \mid Y]) = 0 \mid Y = y] \;=\; 1.$$

Then, to prove almost sure convergence is straightforward by writing

$$\mathbb{P}[\lim_{m\to\infty}(X_j^{(m)} - \mathbb{E}[X_j^{(m)} \mid Y]) = 0] \;=$$

$$= \int \mathbb{P}[\lim_{m\to\infty}(X_j^{(m)} - \mathbb{E}[X_j^{(m)} \mid Y]) = 0 \mid Y = y]dN(y) \;=\; 1.$$

Proposition 6 implies that the conditional expectation $\mathbb{E}[X_j^{(m)} \mid Y]$ for $Y = y$ equals

$$\mathbb{E}[X_j^{(m)} \mid Y = y] \;=\; \frac{1}{E_j^{(m)}}\sum_{i=1}^{m} E_{i,j}\,\mathbb{E}[\mathbf{1}_{\{j-1\leq 4\tau_i<j\}} \mid Y = y] \;=$$

$$\mathbb{P}\Big[\frac{j-1}{4} \leq \tau_i < \frac{j}{4} \mid Y = y\Big] \;=\; N\Big[\frac{N^{-1}(p_{\frac{j}{4}}) - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}\Big] - N\Big[\frac{N^{-1}(p_{\frac{j-1}{4}}) - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}\Big].$$

This completes the proof of the proposition. □

Note that the idea underlying the proof of Proposition 7 does not rely on $Y \sim N(0,1)$. The same argument could be used to establish an analogous convergence result for other than normal distributions. We now come to the final result of this section.

Corollary 1 *Under the conditions stated in this section, we have*

$$\mathbb{P}\Big[\lim_{m\to\infty}\Big(L^{(m)} - \sum_{j=1}^{T} w_j\, g_j(Y)\Big) = 0\Big] \;=\; 1,$$

where the functions $g_j(\cdot)$ *are defined on* $\mathbb{R}$ *by*

$$g_j(y) \;=\; N\Big[\frac{N^{-1}(p_{\frac{j}{4}}) - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}\Big] - N\Big[\frac{N^{-1}(p_{\frac{j-1}{4}}) - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}\Big].$$

The numbers w_j *refer to the limit exposure weights from Condition 2.*

Proof. The assertion follows from the previous two propositions and Condition 2. □

The semi-analytic technique can be further developed and refined in practice in several directions, for example, stochastic recoveries could be implemented into the framework quite easily; see [13], pages 86-89, for an approach for a single-period model, which can be extended to a multi-period approach in a straightforward manner.

From a CDO modeling point of view, the tool developed in this section is quite powerful, because in contrast to the purely analytic approach explained in Section 4.4, the semi-analytic approach allows for the implementation of all relevant cash flow elements, e.g., redirection of cash flows due to realized losses or other ‘triggers’ affecting the performance of CDO notes. This flexibility is a consequence of considering every single payment period, such that all the specialties of the considered waterfall can be implemented in an accurate manner.

In the next section we apply the semi-analytic technique to an illustrative sample transaction.

5 Further examples and applications

We conclude this paper by two examples. Here, we keep the examples as simple as possible and do not exercise the modeling of some complicated structure. More sophisticated examples and illustrations can be found in [14].

An example of an evaluation of a collateralized swap obligation

The first example is a plain-vanilla collateralized swap obligation (CSO) with 5 tranches. The transaction is assumed to work as follows:

- The issuer sells protection by means of 80 single credit default swaps (CDS) with a total swap volume of 800 million EUR (10 million swap volume on each of the 80 names). This constitutes the asset side of the transaction.

- The issuer also buys protection on the 800 million credit volume he now is exposed to. This builds-up the liability side of the structure.

- For buying protection, the vehicle issues 4 credit-linked notes (CLNs), namely class A, class B, class C and equity. The total volume of notes issued in the capital market equals 100 million EUR (12.5% on the swap volume of 800 million). The cash received from issuing the CLNs is invested in an account of risk-free (cash-equivalent) collateral.

- For the upper 87.5% (700 million), the issuers enters into a super senior swap agreement with an OECD bank in order to buy protection against tail events.

- In case of a default in the CDS pool, the realized loss is paid on the protection selling agreement by liquidating collateral and using the proceeds to make the contingent payments. If losses exceed the 100 million funded volume, the super senior swap counterparty will have to pay for the residual losses not already covered by the available collateral. On the liability side, losses eat into the tranches 'bottom-up' in a way that the total funded volume always matches the amount in the risk-free collateral account. Recovered amounts are reflected on the liability side by a 'top-down' deleveraging of the outstanding swap volume.

- The average rating of the CDS pool is a BBB+, the average PD equals 20bps, the assumed recovery is 34% and the transaction matures in 5 years.

We now calculate for this transaction the PD, EL and LGD. Table 7 shows the result of the Monte Carlo simulation. For the calculation, we used a correlated default times approach based on a normal copula as well as a t-copula with

- an average asset correlation of 20% (linear correlation) in both cases and

- 5 degrees of freedom in the t-copula case;

see Section 4.3 as a reference for copulas and default times. Figure 12 illustrates the impact of a copula change; the result could already be guessed from the earlier presented Figure 9.

Because the t-copula generates a much stronger tail dependency (see Figure 9) for the joint distribution of marginal default times (calibrated from our credit curves from Section 4.1), joint defaults occur more often in the t-copula than in the Gaussian copula case.

	Equity	Class C	Class B	Class A	Super Senior
Vol. [USD]	36,000,000	16,000,000	18,000,000	30,000,000	700,000,000
Vol. [%]	4.50%	2.00%	2.25%	3.75%	87.5%
Rating	NR	BBB	A	AA	AAA
Maturity	Sep 2008	Sep 2008	Sep 2008	Sep 2008	Sep 2008
Spreads	0.00%	4.00%	2.00%	0.40%	0.15%
Normal Copula					
EDF (cumul.)	58.55%	6.29%	2.13%	1.11%	0.23%
EL (cumul.) [%]	25.63%	4.17%	1.62%	0.55%	0.01%
EL (cumul.) EUR]	9,226,290	667,100	291,870	164,170	50,570
LGD	43.77%	66.29%	76.27%	49.39%	3.09%
T-Copula					
EDF (cumul.)	39.64%	8.43%	4.70%	3.39%	1.62%
EL (cumul.) [%]	19.90%	6.68%	4.07%	2.35%	0.12%
EL (cumul.) EUR]	7,165,480	1,069,370	732,760	705,210	813,370
LGD	50.21%	79.30%	86.54%	69.30%	7.15%

Table 7: A sample CSO (illustrative!)

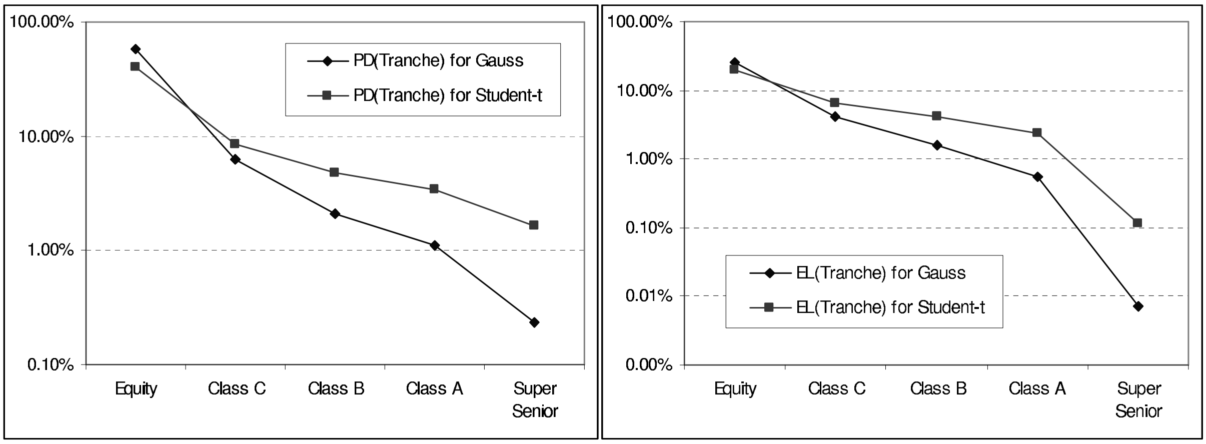

Figure 12: Graphical illustration of PDs (left) and ELs (right) from Table 7

Table 7 and Chart 12 illustrate how the change from a Gaussian to a t-copula with suitably low degrees of freedom *stresses senior tranches* and - at the same time - implies kind of a *risk relief for the most junior tranche*. This is a comparable result to the well-known fact that, e.g., *assuming zero correlation for a first-to-default basket is a conservative approach*; see Figure 14.

The higher the correlation and tail dependency, the better for equity investors and first-to-default takers. The lower the correlation and the more independent the occurrence of joint defaults, the higher the risk of taking the first loss of a basket or pool.

We conclude our example by modeling the transaction also by means of the (semi-)analytic[2] approach. Hereby we rely on a Gaussian copula, but other copulas can be implemented easily. Table 8 shows the result.

[2]Going back to Corollary 1, we see that if $w_j = 1 \; \forall j$ then the semi-analytic approach and the analytic approach are essentially the same for a fixed horizon T. The difference between analytic and semi-analytic approaches in this case is that the 'telescope sum' representation of the cumulative loss according to Corollary 1 allows for all kinds of cash flow adjustments in every single payment period. Obviously, if the pool's exposure profile is not of bullet-type but follows a certain amortization schedule, then the semi-analytic approach and the analytic approach diverge, although in some cases a suitable WAL-adjustment of the analytic approach has some chance to yield good approximations.

	Equity	Class C	Class B	Class A	SSS
PD	100.00%	5.56%	2.32%	0.96%	0.25%
EL	26.98%	3.68%	1.53%	0.52%	0.01%

Table 8: Revisiting the transaction by means of a semi-analytic approach

Here are some comments on the result:

- The PD for equity equals 100%, reflecting the absolute continuity of the analytic loss distribution. We already discussed this phenomenon in the illustrative example in Section 4.4.
- The differences we see are essentially due to the difference of a simulation of 80 non-homogeneous single default times compared to a (semi-)analytic approach based on the assumption of an infinitely granular and homogeneous pool of assets. However, if we would increase the number of assets and take care that the names underlying the CDSs can be modeled by application of a uniform PD and a uniform correlation, then the two results (Table 7 and 8) would converge towards a common risk profile of tranches.

In [14] we also consider pricing and return aspects of a sample CDO. In transactions where, e.g., an excess spread redirection trigger (or, to mention another example, principal deficiency ledgers) are implemented, the semi-analytic approach really unfolds its strengths.

The importance of a reasonable modeling of the timing of defaults

In this section we want to make the point that an accurate modeling of the *timing* of defaults can be quite essential. To give an example, we present Figure 13.

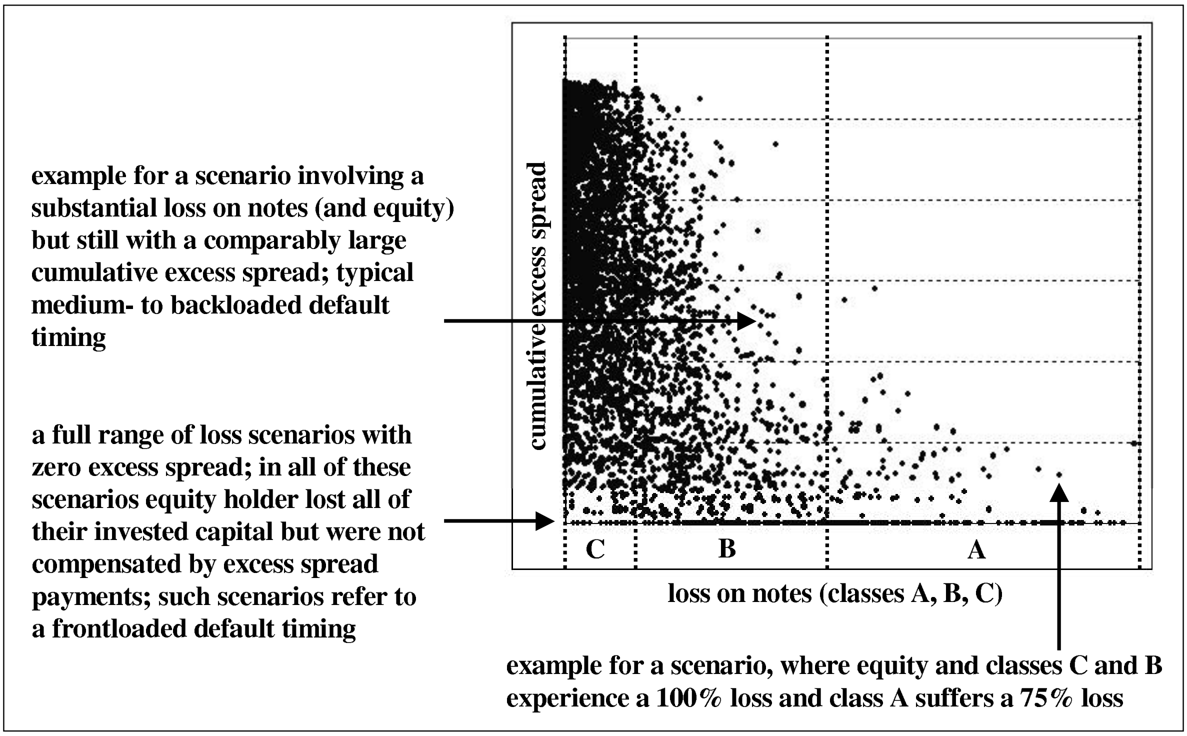

Figure 13: Illustration of the importance of an 'accurate' modeling of *default timing*

The chart shows a scatterplot of loss (x-axis) versus cumulative excess spread (y-axis) paid to equity investors. The underlying transaction has 5 tranches: 4 funded tranches (equity, classes C, B, A) and an unfunded super senior tranche. The loss axis in Figure 13 refers to losses cumulating on classes C, B and A. Based on subordination, any loss eating into class C implies that equity investors already lost all their invested capital. It is interesting to observe the following:

- There are scenarios involving a large loss and nevertheless a substantially high excess spread cumulation to subordinated note holders. The reason for such scenarios typically is a more backloaded default timing.

- There are also scenarios with a comparably small loss but nevertheless a negligibly small excess spread cumulation to equity investors. Such scenarios are generated by a more front-loaded default timing.

- The density of points decreases with increasing seniority of tranches. This reflects the shape of the loss distribution of the underlying asset pool. However, it is worthwhile to mention that losses located at the upper boundary of the C-tranche admit the whole spectrum of excess spread scenarios, ranging from zero to almost maximum excess spread cumulation to equity holders. Mezzanine tranches bear the risk of the second loss but have no participation in the excess spread of the transaction. Therefore they need an extra careful evaluation and risk/return assessment.

Altogether one clearly can see that an accurate modeling of default *timing* is essential for pricing a CDO tranche. To give another example in this direction, Figure 14 shows first-to-default distributions for the transaction described in Table 7.

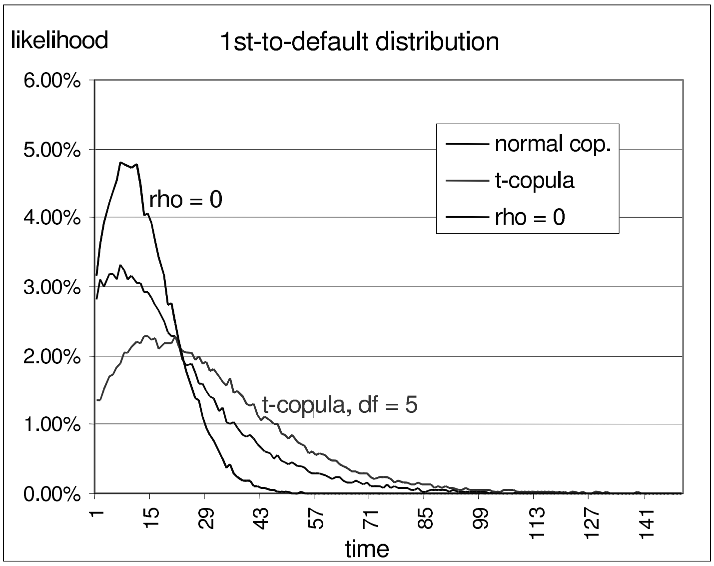

Figure 14: First-to-default distributions

What we see here is that a change of copula as well as a change of linear correlation changes the *time dynamics* of default times. A typical application of default timing considerations are the tailor-made definition of coverage ratios in cash flow CDOs. The more precise an arranger/originator can predict the timing of defaults, the more target-oriented the structuring of the deal can be done.

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